Eliminating Envy in Multigraphs Through Subsidy

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Abstract

Envy-free (EF) allocation is a fundamental problem at the intersection of theoretical computer science and economics. When resources are indivisible, achieving an EF allocation is often impossible. A common remedy is to compensate agents with subsidies. Brustle et al. proved that a total subsidy of $n-1$ is necessary and sufficient to guarantee EF in the worst case, where n is the number of agents, and each agent has additive valuations with every item's maximum marginal value set to 1. In this paper, we consider the graph orientation setting that was recently introduced by Christodoulou et al. In this model, agents correspond to vertices in a multigraph, and resources correspond to edges, with each edge allocated to one of its two incident agents. Despite extensive study of the graph orientation model, it remains unclear how much subsidy is sufficient to guarantee EF. We show that a total subsidy of $\frac{n}{2}$ is always sufficient to guarantee EF in any multigraph, halving the subsidy required in the unconstrained setting, and provide a polynomial-time algorithm to compute such an allocation. We show that this bound is tight, even for simple graphs.

1 Introduction

Fair allocation of indivisible resources is an active area of research in computer science, economics, and operations research. One of the most widely accepted fairness notions is envy-freeness (EF), which requires that no agent strictly prefers another agent's allocation over their own [Foley, 1967]. Unlike the case of divisible items, where EF allocations always exist [Brams and Taylor, 1995], an EF allocation is not guaranteed when the items are indivisible. Consider a simple example: allocating a single item between two agents, each valuing the item at \$1. The agent who receives the item would inevitably be envied by the other. To maintain fairness, a compensation of \$1 can be given to the agent who receives nothing. More generally, this motivates the compelling and practical research problem of understanding how much compensation (i.e., subsidy) is required to ensure envy-freeness among the agents. This problem has been widely studied in the literature. Halpern and Shah [2019] first proved that when agents have additive valuations and the marginal value of each good is at most 1, a total subsidy of $m(n-1)$ is sufficient to find an EF allocation. This result was later improved to $n-1$ by Brustle et al. [2020]. This bound is tight in the sense that there are instances that require at least $n-1$ subsidy in order to achieve EF.

In this paper, we study a graph orientation model first introduced by Christodoulou et al. [2023], which has since attracted considerable attention. Here, agents are the vertices in a multigraph and items are the edges so that each agent is only interested in the items incident to her. An allocation is said to be an *orientation* if every item is allocated to one of the two incident agents. Christodoulou et al. [2023] considered a relaxed definition of envy-freeness, envy-free up to any item (EFX), and proved that deciding the existence of an EFX orientation is NP-hard. An allocation is EFX if no agent envies another agent's bundle after (hypothetically) removing any item in it. There is a significant amount of follow-up work on the graph orientation problem, including characterizing the graph structures under which an EFX orientation exists [Zeng and Mehta, 2024; Blazej *et al.*, 2025], designing approximation algorithms for EFX [Amanatidis *et al.*, 2024; Kaviani *et al.*, 2025], and exploring other (approximate) fairness notions [Deligkas *et al.*, 2025; Christodoulou and Mastrakoulis, 2025]. However, it remains unclear how much subsidy is sufficient to ensure EF in the graph orientation model. In this work, we address the problem of bounding the worst-case subsidy required to ensure envy-freeness in the graph orientation model.

1.1 Our Problem

Let $G = (V, E)$ be a multigraph, where $V = \{1, \dots, n\}$ and $E = \{e_1, \dots, e_m\}$ are respectively the sets of vertices and edges with $|V| = n$ and $|E| = m$. There may be multiple edges between any two agents; however, without loss of generality, we assume the absence of self-loops or singletons, as self-loops and singletons can only serve to reduce envy among agents. Let $E_{i,j}$ denote the set of edges between $i, j \in V$, where $E_{i,j} = E_{j,i}$ and $E_{i,j} = \emptyset$ if i, j are not adjacent in G. Let $E_i = \bigcup_{j \in V} E_{i,j}$ be the set of edges incident to $i \in V$.

In the fair graph orientation model, each vertex $i \in V$ corresponds to an agent, and each edge $e \in E$ corresponds to a good (or item). We also denote the sets of agents and goods by $N = V$ and $M = E$ to be consistent with the literature on resource allocation, and thus we use agents and vertices, goods and edges, interchangeably. Each agent i has an additive valuation function $v_i(\cdot) : 2^M \to \mathbb{R}_{\geq 0}$ such that for any $S \subseteq M$, $v_i(S) = \sum_{e \in S} v_i(\{e\})$. For simplicity, denote $v_i(\{e\})$ by $v_i(e)$. In the orientation model, each agent $i \in V$ is only interested in her incident goods E_i, which are also called i's relevant goods, and each good can only be allocated to one of the two incident agents. That is $v_i(e) = 0$ for all $e \notin E_i$, and a feasible allocation, called *orientation*, is an n-partition $\mathbf{A} = (A_1, \ldots, A_n)$ of M such that $A_i \subseteq E_i$ for all i. We normalize each valuation such that $\max_{e \in M} v_i(e) = 1$ for all $i \in N$.

With no subsidies, an allocation $\mathbf{A}$ is *envy-free* (EF) if no agent would prefer the bundles of others to her own bundle, i.e., $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$. As mentioned, when items are indivisible, an EF allocation barely exists. To address this, we provide each agent i with an objective payment (i.e., money) $p_i \geq 0$, called *subsidy*, alongside their allocated goods. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be the subsidy vector, and denote by $(\mathbf{A}, \mathbf{p})$ the final outcome. Accordingly, an allocation with subsidy $(\mathbf{A}, \mathbf{p})$ is envy-free if for any $i, j \in N$, $v_i(A_i) + p_i \geq v_i(A_j) + p_j$.

To illustrate, let's consider the following example.

Example 1. *Assume n is even, and consider a simple graph with vertices $\{1, \ldots, n\}$. For any $i \in \{1, \ldots, \frac{n}{2}\}$, there exists an edge between vertices $2i - 1$ and $2i$, and hence, in total $\frac{n}{2}$ edges. Each edge is valued at 1 by both of its incident agents. We can allocate each edge to one of its incident agents and provide the other agent with a subsidy of 1. This results in an EF orientation with a total subsidy of $\frac{n}{2}$.*

Next, assume n is odd, the simple graph contains a component consisting of a path of length 2, and $\frac{n-3}{2}$ components, each of which is a path with length 1. Each agent has a value of 1 for any of her incident edges. To obtain an EF orientation, we similarly allocate every edge to one incident agent such that no agent receives more than one edge, and provide each agent who has not received edges a subsidy of 1. The total subsidy is $\frac{n-1}{2}$.

It is worth noting that, in Example 1, a total subsidy of less than $\lfloor \frac{n}{2} \rfloor$ is insufficient to guarantee EF for any orientation. In other words, in the worst case, a total subsidy of $\lfloor \frac{n}{2} \rfloor$ is necessary. This naturally leads to the question of whether this bound is also sufficient:

> *Is a total subsidy of $\lfloor \frac{n}{2} \rfloor$ sufficient to ensure EF for all multigraphs?*

The main contribution of the current paper is to answer the above question in the affirmative. Specifically, we present a polynomial-time algorithm that always computes an EF orientation with a total subsidy of at most $\lfloor \frac{n}{2} \rfloor$. An overview of our algorithm is provided in Section 2, and a formal proof of our result is given in Section 3. We note that the normalization assumption $\max_{e \in M} v_i(e) = 1$ for all $i \in N$ is essential for our result. If this assumption is relaxed to $\max_{e \in M} v_i(e) \leq 1$, as in [Halpern and Shah, 2019; Brustle *et al.*, 2020], the optimal subsidy bound remains $n-1$. For example, consider a graph with a single edge valued at 1 by both of its incident agents, while all other agents correspond to singleton vertices. In this case, a subsidy of $n - 1$ is required: one incident agent receives the edge, and each of the remaining agents receives a unit subsidy.

1.2 Related Work

Since EF allocations may not exist, the majority of the literature is focused on its relaxations when subsidy is not allowed. Two widely adopted relaxations are envy-free up to one item (EF1) [Budish, 2011] and envy-free up to any item (EFX) [Caragiannis *et al.*, 2019], which requires no agent strictly prefers another agent's bundle excluding one or any item. EF1 allocations always exist [Lipton *et al.*, 2004] but the existence of EFX allocations is still unknown except for several special cases, e.g., [Plaut and Roughgarden, 2020; Chaudhury *et al.*, 2024]. We refer to the recent surveys [Amanatidis *et al.*, 2023; Liu *et al.*, 2024] for a comprehensive overview of fair allocation. Below we restrict our focus on the graph orientation and fair division with subsidy.

Graph Orientation. Christodoulou et al. [2023] first introduced the graph model where vertices are agents and edges are items such that agents are only interested in their incident items. When agents have additive valuations, they proved that an EFX allocations when the items can be arbitrarily allocated always exists but an EFX orientation where items must be allocated to their incident agents may not exist. Zeng and Mehta [2024] identified a family of graphs in which EFX orientations exist. Kaviani et al. [2024] and Deligkas et al. [2025] repectively generalized the graph orientation problem. In the model of Kaviani et al. [2024], agents have (p, q)-bounded valuations, where each item has a non-zero marginal value for at most p agents, and any two agents share at most q common interested items. They proved that for $(\infty, 1)$-bounded valuations, EF2X orientations always exist. In the model of Deligkas et al. [2025], every agent has a predetermined set of items and only receives items from this set. They proved that for monotone valuations, an EF1 allocation always exists. Zhou et al. [2024] extended the goods model in [Christodoulou *et al.*, 2023] to mixed setting when the marginal value of each item can be positive or negative.

Fair Division with Subsidy. Caragiannis and Ioannidis [2021] first studied the computational complexity of using the minimum amount of subsidies to ensure envy-freeness. Halpern and Shah [2019] bounded the worst-case amount of subsidies for additive valuations, and Brustle et al. [2020] later improved the result in [Halpern and Shah, 2019]. There is a line of recent works focusing on the variants and generalizations of the standard setting. For example, Goko et al. [2024] considered the strategic behaviors when subsidy is used, and Aziz et al. [2024] considered weighted envy-freeness when the agents have different entitlements. Brustle et al. [2020] proved that for general monotone valuations an EF allocation always exists with a subsidy of $2(n-1)^2$, which was improved to $n(n-1)/2$ in [Kawase *et al.*, 2024]. It is worth noting that the tight bound for monotone valuations is still unknown [Liu *et al.*, 2024], where the lower bound is $n - 1$. However, there have been improvements for certain special cases. Barman et al. [2022] and la Tour and Fujii [2025], respectively, improved the upper bounds to $O(n)$ and $O(n\sqrt{n \log n})$ for monotone valuations with binary marginal values and when the number of agents is a prime power. Choo et al. [2024] and Dai et al. [2024] respectively studied the envy-free house allocation with subsidies and its weighted variant. Wu et al. [2023; 2025b; 2025a] extended the subsidy model to the fair allocation of indivisible chores.

2 Technical Overview

In this section, we present the definitions and techniques used to prove our main result. We discuss the challenges involved and outline the key ideas behind our algorithm.

2.1 Envy-freeable Orientations

Before describing our algorithm, it is important to note, as observed by Halpern and Shah [2019], that not every orientation can be made envy-free through the use of subsidies. For example, consider an instance with 2 agents and 2 edges between them such that $v_1(e_1) = v_2(e_2) = 1$ and $v_1(e_2) = v_2(e_1) = 0$. If we allocate $\{e_2\}$ to agent 1 and $\{e_1\}$ to agent 2, no matter how much we compensate each agent, the envy cannot be eliminated, since the subsidies are objective. Thus, Halpern and Shah [2019] introduced the notion of envy-freeable allocations, and characterized all envy-freeable allocations via *envy graphs*. Formally, an allocation $\mathbf{A}$ is said to be *envy-freeable* (EFable) if there exists a subsidy vector $\mathbf{p} = (p_1, \ldots, p_n)$ such that $(\mathbf{A}, \mathbf{p})$ is EF. The envy graph $D_{\mathbf{A}}$ for allocation $\mathbf{A}$ is a complete weighted directed graph with vertices N. For any pair of agents $i, j \in N$, the weight of arc $i \to j$ in $D_{\mathbf{A}}$ is $v_i(A_j) - v_i(A_i)$.

Theorem 1 ([Halpern and Shah, 2019])**.** *For any allocation $\mathbf{A}$, the following statements are equivalent:*

1. *$\mathbf{A}$ is envy-freeable;*
2. *$\mathbf{A}$ maximizes the total value of agents among all reassignments of its bundles to agents, i.e., for every permutation $\pi : N \to N$, $\sum_{i\in N} v_i(A_i) \geq \sum_{i \in N} v_i(A_{\pi(i)})$;*
3. *$D_{\mathbf{A}}$ contains no directed cycle with positive weight.*

Given orientation $\mathbf{A} = (A_1, \ldots, A_n)$, for any $i, j \in N$, denote $A_i^j = A_i \cap E_{i,j}$ for ease of notation. The orientation is *locally envy-freeable* with respect to agents i, j if the partial allocation of $E_{i,j}$ is envy-freeable, i.e., $v_i(A_i^j) + v_j(A_j^i) \geq v_i(A_j^i) + v_j(A_i^j)$. The orientation $\mathbf{A}$ is locally envy-freeable if for every pair of agents i, j, the partial allocation of $E_{i,j}$ is envy-freeable.

2.2 Our Algorithm

Brustle et al. [2020] proposed the "Bounded-Subsidy Algorithm" that computes an EF allocation with subsidy at most $n-1$ for the classical additive allocation setting, where each good may be assigned to any agent rather than being constrained by a graph orientation. However, this algorithm cannot achieve the bound of $\lfloor \frac{n}{2} \rfloor$ in the context of orientation, even for simple graphs. Hence, one needs to derive a new method to achieve this bound.

Since every edge is only relevant to two agents, say i and j, another intuition is to find a proper way to allocate the edges between the two agents and compensate the one at a disadvantage. Two useful and commonly used rules are the Round-Robin algorithm shown in Algorithm 1 and the social utility maximizing algorithm shown in Algorithm 2. Although the other agents are not directly affected by the edges between i and j, they may feel envious of any subsidy provided. Furthermore, this independent procedure does not provide control over the total amount of subsidy given. Before moving on to our algorithm, it is worth noting that in the partial allocation returned by $\mathsf{RoundRobin}(i, j, E_{i,j})$, the envy of one agent towards the other one is upper bounded by 1, the largest single-item value, although this allocation may not be locally envy-freeable. However, by Theorem 1, the partial allocation returned by $\mathsf{MaxUtility}(i, j, E_{i,j})$ is always locally envy-freeable.

Algorithm 1 $\mathsf{RoundRobin}(i, j, S)$

Input: Agents i, j and bundle S.
Output: A 2-partition (B_i, B_j) of S.

$t \leftarrow i$. {Round Robin starts with agent i}
while $S \neq \emptyset$ **do**
  $e^* \leftarrow \arg\max_{e \in S} v_t(e)$, breaking ties arbitrarily when no explicit rule is specified;
  $B_t \leftarrow B_t \cup \{e^*\}$, $S \leftarrow S \setminus \{e^*\}$, and $t \leftarrow \{i, j\} \setminus \{t\}$;
end while

Algorithm 2 $\mathsf{MaxUtility}(i, j, S)$

Input: Agents i, j and bundle S.
Output: A 2-partition (B_i, B_j) of S.

for $e \in S$ **do**
  $t \leftarrow \arg\max_{t \in \{i,j\}} v_t(e)$. If there is a tie, set $t \leftarrow i$;
  $B_t \leftarrow B_t \cup \{e\}$;
end for

We now introduce the high-level ideas of our algorithm. Given an orientation instance $G = (V, E)$ and agents' valuations $v_i(\cdot)$'s, our algorithm utilizes an auxiliary directed graph $G' = (V, E')$ over the agents, called the *reserve graph*. The vertex set of G' is identical to that of G. Each agent i selects, or reserves, one incident edge only for the purpose of constructing the auxiliary graph G': choose an edge $e \in E_i$ such that $v_i(e) = 1$; if there are multiple, we arbitrarily choose one. This selection is not an allocation, and the selected edge may or may not eventually be assigned to agent i. Suppose $e \in E_{i,j}$, then we create a directed edge $j \to i$ in E'. The existence of such an edge follows from the normalization $\max_{e \in E_i} v_i(e) = 1$. If two agents i and j select edges in $E_{i,j}$, possibly the same edge, then both arcs $i \to j$ and $j \to i$ are created in E'. Thus, the in-degree of each vertex is 1 in G', and the out-degree can be any number between 0 and $n - 1$. The intuition behind the reserve graph is that if agent i, who has neighbors j and k in G, eventually receives the edge of value 1 she selected between i and j, then i does not envy k provided that the edges

$E_{i,k}$ are allocated by $\mathsf{RoundRobin}(k, i, E_{i,k})$. Recall that under $\mathsf{RoundRobin}(k, i, E_{i,k})$, i's envy towards k is at most 1, which can be offset by the value of the reserved edge.

Denote by $\mathcal{C} = \{C_1, \ldots, C_k\}$ the set of weakly connected components in G', where each component is a maximal connected subgraph of G' when edge directions are ignored. As each vertex has in-degree 1, each component is either a directed cycle or a set of trees rooted at vertices in a cycle. Our algorithm proceeds in two phases:

- In Phase 1, allocate edges between adjacent agents in G';
- In Phase 2, allocate edges between non-adjacent agents in G'.

Regarding Phase 1, based on the earlier intuition, if one component C does not contain a 2-cycle (a cycle with length 2), we can run $\mathsf{RoundRobin}(j, i, E_{i,j})$ for all pairs of agents such that $i \to j$ in C. This results in a partial allocation that is EF and does not require any subsidy. However, when there is a 2-cycle $f \to g$ and $g \to f$ in C, the problem becomes tricky, as there are no additional reserved edges available to offset the mutual envy between f and g. In such cases, it is necessary to provide a subsidy to one of the agents, say f. While it is not hard to ensure that f's envy towards g remains within 1, the real challenge lies in managing the envy between f and her other neighbors, as well as the envy that propagates to their respective neighbors.

To resolve this issue, we introduce a diffusion subroutine as shown in Algorithm 5, which essentially distributes the subsidy of f to her descendants. Accordingly, we prove an important property that for each component C, the total required subsidy is upper bounded by the envy of f towards g (see Lemma 7), which is upper bounded by 1. In summary, the total subsidy is upper bounded by k, where $k \leq \lfloor \frac{n}{2} \rfloor$ is the number of components in G'. This is the most technically complicated part of the algorithm. Note that this analysis matches Example 1, where the edges form $\lfloor \frac{n}{2} \rfloor$ paths.

In Phase 2, we allocate the edges between non-adjacent agents in G'. To address any resulting envy, we adjust the subsidies provided in Phase 1, potentially reducing the subsidy for certain agents as needed.

3 Main Result

In this section, we formally describe the algorithm and prove the following main result.

Theorem 2. *For any multigraph with n agents, an EF orientation with a total subsidy of at most $\lfloor \frac{n}{2} \rfloor$ can be computed in polynomial time.*

3.1 The Algorithm

The main algorithm is shown in Algorithm 3, with two subroutines described in Algorithms 4 and 5. The algorithm first builds *reserve graph* G', and let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be the k weakly connected components of G'. For simplicity, we use $(x)^+$ to denote $\max(x, 0)$.

Phase 1 (Lines 4–21). In this phase, we allocate items between agents adjacent in G'. For a component C of G', if

Algorithm 3 Main Algorithm for Multigraph Orientations

Input: An instance $G = (N, E)$ and valuation $\{v_i\}$.
Output: An orientation $\mathbf{A} = (A_1, \ldots, A_n)$ and payment $\mathbf{p} = (p_1, \ldots, p_n)$.

```
1: For each i ∈ N, initialize A_i ← ∅, w_i, t_i, p_i ← 0.
2: Create a reserve graph G′ as described in Section 2.2. % The in-degree of each vertex is 1 in G′.
3: Let 𝒞 = {C_1, ..., C_k} be the k weakly connected components of G′.
   % Phases 1: allocate items between adjacent agents in G′
4: for each C ∈ 𝒞 do
5:     if C has a directed cycle of length at least 3 then
6:         for each arc i → j in C do
7:             RoundRobin(j, i, E_{i,j}) = (B_j, B_i) and respectively allocate B_j and B_i to agents j and i.
8:         end for
9:     else
10:        Suppose agents f, g are the (only) two agents such that arcs f → g and g → f exist in C, and then execute RoundRobin(g, f, E_{f,g}). If the resulting partial allocation is not locally EFable, swap their partial bundles restricted to E_{f,g}.
11:        if both f and g are envy-free when restricted to the allocation of E_{f,g} then
12:            for i = f, g do
13:                Start from i and form an out-tree (excluding {f, g} \ {i}) with root i. Recursively execute Subroutine1 to allocate items between each node and its children, proceeding level by level down the tree until reaching the leaves.
14:            end for
15:        else
16:            Suppose that agent g does not envy f when restricted to the allocation of E_{f,g};
17:            Start from f and form an out-tree (excluding g) with root f. Recursively use Subroutine2 to allocate items between each node and its children, proceeding level by level down the tree until reaching the leaves.
18:            Start from g and form an out-tree (excluding g) with root g. Recursively use Subroutine1 to allocate items between each node and its children, proceeding level by level down the tree until reaching the leaves.
19:        end if
20:    end if
21: end for
    % Phases 2: allocate items between non-adjacent agents
22: For each i ∈ N, p_i ← t_i.
23: for each set of edges E_{i,j} not allocated do
24:     Execute RoundRobin(j, i, E_{i,j}). If the resulting partial allocation is not locally EFable when restricted to E_{i,j}, swap their partial bundles restricted to E_{i,j}.
25:     Let j be the agent such that v_j(A_j^i) ≥ v_j(A_i^j)
26:     if v_i(A_j^i) + p_j > v_i(A_i) + p_i then
27:         p_j ← (v_i(A_i) + p_i − v_i(A_j^i))^+.
28:     end if
29: end for
```

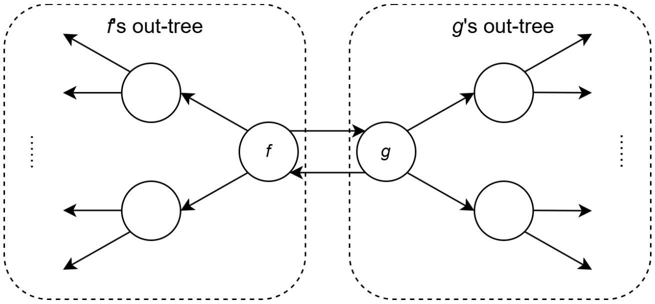

Figure 1: An illustration of the component with a 2-cycle and the out-trees of f and g.

the directed cycle has length at least 3, use RoundRobin (see Algorithm 1) to allocate items between each pair of adjacent agents i, j in C. In particular, if $i \to j$ exists, let j pick first in RoundRobin, and if $j \to i$ exists, let i pick first.

For the other case where C has a directed cycle with length 2 or a directed 2-cycle, suppose that both arcs $f \to g$ and $g \to f$ exist. First, allocate $E_{f,g}$ through RoundRobin. If the resulting allocation is not locally EFable with respect to f, g, then swap the bundles (Line 10) to ensure local EFability. Next, form out-trees of f and g (see Figure 1 for an illustration) and proceed with the allocation of each out-tree from the corresponding root to depth-1 vertices first. Here, the depth of a vertex is measured from the root of the corresponding out-tree: the root has depth 0, and the depth-1 vertices are exactly the children/out-neighbors of the root in that out-tree.

Suppose that the algorithm visits f, of whom the set of out-neighbors is $Ch(f)$ (excluding g) in the out-tree. For each $j \in Ch(f)$, items in $E_{f,j}$ are allocated based on either Subroutine1 or Subroutine2, depending on the allocations of $E_{f,g}$ (Lines 11–18). The idea of Subroutine1 is to allocate items in a round robin manner, while that of Subroutine2 is relatively more complicated and will be explained later. These two subroutines are formally described in Algorithms 4 and 5. After allocating items between f and its depth-1 vertices, i.e., the children/out-neighbors of f in the out-tree, the algorithm visits each depth-1 vertex in the out-tree of f, and allocates items between these vertices and their adjacent depth-2 vertices, following the same rule used for allocating items incident to f. Repeat the process until the leaves (vertices with out-degree 0) are reached. This allocation rule also applies to the vertices in the out-tree of g. At the completion of this phase, items between agents adjacent in G' have been allocated.

Phase 2 (Lines 21–29). Allocate items between agents not adjacent in G'. For each pair of i, j that are adjacent in G but not in G', items $E_{i,j}$ are currently unallocated. Allocate $E_{i,j}$ through subroutine RoundRobin. If the resulting partial allocation is not locally EFable with respect to i, j, then swap their bundles restricted to $E_{i,j}$. Finally, Lines 26 and 27 adjust the payment to ensure that no envy is created between i, j.

Next, we introduce the Subroutine1 and Subroutine2, which allocate items between agents within the same component of reserve graph G'.

Subroutine1. It takes an agent i as an input and allocate items between i and her out-neighbors in G'. In particular, for each out-neighbor j, allocate items in $E_{i,j}$ through RoundRobin and let j pick first. This subroutine is formally introduced in Algorithm 4.

Algorithm 4 Subroutine1(i)

Input: An agent i.
$Ch(i) \leftarrow$ the set of i's out-neighbors in G', excluding i's in-neighbor if applicable.
for each $j \in Ch(i)$ **do**
  Let RoundRobin$(j, i, E_{i,j}) = (B_j, B_i)$ and allocate respectively B_j and B_i to agents j and i;
end for

Subroutine2. It also takes an agent i as an input and allocate items between i and her out-neighbors in G', with a rule different from Subroutine1. Let $Ch(i)$ be the set of out-neighbors of i in G' and s be the in-neighbor of i. According to the main algorithm, when Subroutine2 visits i, items in $E_{i,s}$ have already been allocated.

When visiting i, the subroutine first updates w_i based on the allocation $E_{i,s}$ and t_s (Line 4). Note that t_s was determined when Subroutine2 visits s. The intuition behind w_i is that if agent i receives subsidy w_i and agent s receives subsidy t_s, then agent i does not envy agent s when restricted to $E_{i,s}$. Then for each $j \in Ch(i)$, compute the pre-assignments of $E_{i,j}$ through RoundRobin (Lines 5–7). In Line 8, by the pre-assignments, we find the agent k, and define $R \subseteq Ch(i)$ based on the sum of agent k's value in the pre-assignments and t_k (Line 9). For agent $j \in R$, we compute another pre-assignment of $E_{i,j}$ through MaxUtility.

Next, classify agents in R into Q_1', Q_1, Q_2, Q_3 based on w_i and the envy of j towards i in the pre-assignment determined by MaxUtility. Then allocate items between i and these agents (Lines 13–25). The items between i and agents in $Ch(i) \setminus R$ are allocated in Lines 30–34. Finally, we update t_i in Line 35. Note that in later rounds, t_i will be used to update w_j for every out-neighbour j of i. As we will see, both w_i and t_i serve as the upper bounds of the final payment for agent i, and moreover, $t_i \le w_i$.

Figure 2 illustrates the classification of $Ch(i)$, and the allocation algorithms used between i and agents in each group. Note that we process Q_4 differently from other groups using RoundRobin, where we additionally swap the bundles between i and the agent in Q_4 returned by RoundRobin.

Remark that for each i, w_i (Line 4 of Subroutine2) is updated before allocating the items between i and i's out-neighbors while t_i is updated after all items between i and i's out-neighbors are allocated. Both w_i and t_i will be updated (at most) once during the process.

3.2 Proof of Theorem 2

In this section, we show that Algorithm 3 computes an envy-free orientation and payment $(\mathbf{A}, \mathbf{p})$ with $\sum_{i \in N} p_i \le \lfloor \frac{n}{2} \rfloor$.

Algorithm 5 Subroutine2(i)

Input: An agent i.
1: Initialize $R, Q_1, Q'_1, Q_2, Q_3, Q_4, Q_5 \leftarrow \emptyset$ and $k \leftarrow 0$.
2: $s \leftarrow$ the in-neighbor of i in G'.
3: $Ch(i) \leftarrow$ the set of i's out-neighbor in G', excluding s (if applicable).
4: Define $w_i := (t_s + v_i(A_s^i) - v_i(A_i^s))^+$.
5: **for** every $j \in Ch(i)$ **do**
6:     Let RoundRobin$(j, i, E_{i,j}) = (T_j^i, T_i^j)$ where T_i^j and T_j^i respectively are the bundles for agent i and j. When the underlying agent has value 1 for multiple items, the agent always selects the item resulting in the edge in G'.
7: **end for**
8: $T_s^i \leftarrow A_s^i$ and $k \leftarrow \arg\max_{j \in Ch(i) \cup \{s\}} (v_i(T_j^i) + t_j)$ % Here $t_j = 0$ for all $j \in Ch(i)$.
9: **for** $j \in Ch(i)$ with $v_i(E_{i,j}) < v_i(T_k^i) + t_k$ **do**
10:     Let MaxUtility$(j, i, E_{i,j}) = (S_j^i, S_i^j)$ where S_i^j and S_j^i respectively are bundles corresponding to agents i and j.
11:     $R \leftarrow R \cup \{j\}$.
12: **end for**

....

13: **for** $j \in R$ with $0 \leq v_j(S_i^j) - v_j(S_j^i) \leq w_i$ **do**
14:     **if** $Q_1 = \emptyset$ **then**
15:         $Q_1 \leftarrow Q_1 \cup \{j\}$, $A_j \leftarrow A_j \cup S_j^i$, and $A_i \leftarrow A_i \cup S_i^j$.
16:     **else**
17:         $Q'_1 \leftarrow Q'_1 \cup \{j\}$, $A_j \leftarrow A_j \cup T_j^i$, and $A_i \leftarrow A_i \cup T_i^j$.
18:     **end if**
19: **end for**

....

20: **for** $j \in R$ with $w_i < v_j(S_i^j) - v_j(S_j^i)$ **do**
21:     $Q_2 \leftarrow Q_2 \cup \{j\}$, $A_j \leftarrow A_j \cup T_j^i$, and $A_i \leftarrow A_i \cup T_i^j$.
22: **end for**

....

23: **for** $j \in R$ but does not satisfy the conditions of Line 13 or 20 **do**
24:     $Q_3 \leftarrow Q_3 \cup \{j\}$, $A_j \leftarrow A_j \cup S_j^i$, and $A_i \leftarrow A_i \cup S_i^j$.
25: **end for**

....

26: **for** $j \notin R$ **do**
27:     $A_j \leftarrow A_j \cup T_j^i$ and $A_i \leftarrow A_i \cup T_i^j$.
28: **end for**
29: $S \leftarrow \{j \in Ch(i) \text{ and } j \notin R : v_i(A_j^i) + v_j(A_i^j) > v_i(A_i) + v_j(A_j^i)\}$.
30: **if** $S \neq \emptyset$ **then**
31:     $q \leftarrow \arg\max_{j \in S} v_i(A_j^i)$ and break ties arbitrarily. Let $Q_4 \leftarrow \{q\}$.
32:     Swap A_i^q and A_q^i and update A_i, A_q.
33: **end if**

....

34: Define $Q_5 := Ch(i) \setminus (Q'_1 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4)$.

....

35: Update $t_i \leftarrow (v_i(A_k^i) + t_k - v_i(A_i))^+$.

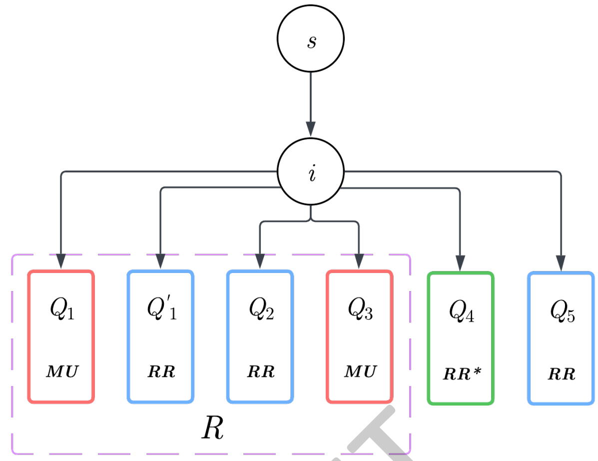

Figure 2: An illustration of the classification of $Ch(i)$ in Subroutine2(i), where MU and RR mean algorithms MaxUtility and RoundRobin, and RR^* means swapping the allocations of RR.

Due to space limitations, the detailed proofs of several lemmas are omitted. We begin with observations and propositions regarding Algorithm 3 and subroutines.

Observation 3. *In the returned allocation* $\mathbf{A}$*, for a pair of agents i and j adjacent in G, bundles A_i^j and A_j^i are determined by either* RoundRobin *or* MaxUtility.

If the allocation of $E_{i,j}$ is determined by RoundRobin, then the envy of any of these two agents towards the other is at most 1.

Proposition 4. *For i, j adjacent in G, suppose* RoundRobin$(j, i, E_{i,j}) = (B_j, B_i)$*, then $v_j(B_j) \geq v_j(B_i)$ and $v_i(B_j) - v_i(B_i) \leq 1$ hold. If respectively assigning B_j and B_i to agents j and i does not result in a locally EFable allocation, then $v_j(B_j) - v_j(B_i) \leq 1$.*

Proof. For the first part of the statement, since agent j picks first, she does not envy agent i, i.e., $v_j(B_j) \geq v_j(B_i)$. For agent i, as RoundRobin presents an EF1 allocation and the maximum marginal value of an item is at most 1, we have $v_i(B_j) - v_i(B_i) \leq 1$.

For the second part of the statement, if allocating B_j (resp., B_i) to j (resp., i) is not locally envy-freeable, then $v_i(B_j) + v_j(B_i) > v_i(B_i) + v_j(B_j)$, implying $v_j(B_j) - v_j(B_i) < v_i(B_j) - v_i(B_i) \leq 1$. □

The tie-break rule in MaxUtility favors one agent when some item is equally valued, and we call this agent the favored agent. If the allocation of $E_{i,j}$ is determined by MaxUtility and the favored agent values some item in $E_{i,j}$ at 1, then the favored agent receives a value of at least 1 from items $E_{i,j}$. The following proposition directly follows from the rule of MaxUtility.

Proposition 5. *For i, j adjacent in G, suppose* MaxUtility$(j, i, E_{i,j}) = (B_j, B_i)$*. If there exists an item $e \in E_{i,j}$ with $v_j(e) = 1$, then $v_j(B_j) \geq 1$.*

Next, we present bounds of w_i, t_i (after updating) if i is visited by Subroutine2.

Proposition 6. *For an agent i, if the items between i and i's out-neighbors in G' are allocated by* Subroutine2*, then i has an in-neighbor s in G'. Moreover, $v_i(A_i^s) + w_i \geq 1$ holds, and after updating t_i in Line 35 of* Subroutine2*, $t_i \leq w_i$.*

Proof. The fact that i has an in-neighbor s in G' directly follows from the construction of the reserve graph. Since arc $s \to i$ exists in G', according to Line 2 of Algorithm 3, there exists $e \in E_{s,i}$ such that $v_i(e) = 1$. Thus, $\max(v_i(A_i^s), v_i(A_s^i)) \geq 1$ holds as $\{A_i^s, A_s^i\}$ is a 2-partition of $E_{s,i}$. Then we have

$$\begin{aligned} w_i + v_i(A_i^s) &= \left(t_s + v_i(A_s^i) - v_i(A_i^s)\right)^+ + v_i(A_i^s) \\ &\geq \max(v_i(A_i^s), v_i(A_s^i)) \geq 1, \end{aligned}$$

where the first inequality transition is due to $t_s \geq 0$.

For the bound between t_i and w_i, if $t_i = 0$, then $t_i \leq w_i$ holds, as $w_i \geq 0$. For the case where $t_i > 0$, we have $t_i = t_k + v_i(A_k^i) - v_i(A_i)$ for some k belonging to $Ch(i) \cup \{s\}$; recall that $Ch(i)$ is the set of out-neighbors (excluding s if applicable) of i in G'. If $k = s$, then since $A_i^s \subseteq A_i$ and t_s, A_s^i are identical in Lines 4 and 35 of Subroutine2, we have $t_i \leq w_i$. If $k \neq s$, at the moment when computing t_i in Line 35, $t_k = 0$. By $k \neq s$, we have $A_i^k \cup A_i^s \subseteq A_i$ and the following holds,

$$\begin{aligned} t_i = t_k + v_i(A_k^i) - v_i(A_i) &\leq v_i(A_k^i) - v_i(A_i^k) - v_i(A_i^s) \\ &\leq 1 - v_i(A_i^s) \leq w_i. \end{aligned}$$

The first inequality comes from $t_k = 0$ and $A_i^k \cup A_i^s \subseteq A_i$. For the second one, since $k \in Q_4 \cup Q_5$ (k is used to define R and $k \notin R$, so $k \in Ch(i) \setminus R = Q_4 \cup Q_5$), then A_i^k and A_k^i are determined by RoundRobin$(k, i, E_{i,k})$. Based on Proposition 4, $v_i(A_k^i) - v_i(A_i^k) \leq 1$. The third one is due to $v_i(A_i^s) + w_i \geq 1$. □

We now present a crucial lemma for bounding the total required subsidy. Recall that $Ch(i)$ denotes out-neighbors of i in G', excluding i's in-neighbor if applicable.

Lemma 7. *If the items between i and $Ch(i)$ are allocated by* Subroutine2*, then at termination of Algorithm 3, it holds that $t_i + \sum_{j \in Ch(i)} w_j \leq w_i$.*

To prove this lemma, we establish key properties guaranteed by Subroutine2 for $Ch(i)$.

Lemma 8. *For each weakly connected component C in G', let N_C be the set of vertices in C. Then it holds that $\sum_{j \in N_C} t_j \leq 1$ at termination of Algorithm 3.*

Proof. If items between adjacent agents in C are all allocated by Subroutine1, then for each $j \in N_c$, $t_j = 0$ holds at termination of Algorithm 3. Thus, we can focus on the case where some items between adjacent agents in C are allocated by Subroutine2. Then there must be a directed 2-cycle in C, and suppose that the 2-cycle is formed by f, g. Moreover, there must be envy between f, g when restricted to $E_{f,g}$. Without loss of generality, assume that f envies g when restricted to $E_{f,g}$. Therefore, items between adjacent agents in the out-tree with root f are allocated by Subroutine2.

Let N_f denote the set of vertices in the out-tree with root f (and hence $g \notin N_f$). By Lemma 7, for each $j \in N_f$, it holds that $t_j \leq w_j - \sum_{\ell \in Ch(j)} w_\ell$. Then we have

$$\begin{aligned} \sum_{j \in N_C} t_j &= t_f + \sum_{j \in N_f \setminus \{f\}} t_j \\ &\leq t_f + \sum_{j \in N_f \setminus \{f\}} (w_j - \sum_{\ell \in Ch(j)} w_\ell) \\ &= t_f + \sum_{\ell \in Ch(f)} w_\ell \\ &\leq w_f. \end{aligned}$$

In the proof of Lemma 7 (which can be found in the full version), we have proved that $w_f \leq 1$ if the items between adjacent agents in the out-tree with root f are allocated by Subroutine2. Therefore, $\sum_{j \in N_C} t_j \leq 1$ holds. □

At the termination of Algorithm 3, each agent j receives bundle A_j and payment p_j. As p_j is never increased during Phase 2 of Algorithm 3, we have $p_j \leq t_j$ for every j. Thus, $\sum_{j \in N} p_j \leq \sum_{j \in N} t_j$. As each vertex has in-degree 1 in G', there are at most $\lfloor \frac{n}{2} \rfloor$ weakly connected components. By Lemma 8, for each such component C in G', $\sum_{j \in N_C} t_j \leq 1$, where N_C is the set of vertices in C. Thus, we have $\sum_{j \in N} t_j \leq \lfloor \frac{n}{2} \rfloor$, which implies $\sum_{j \in N} p_j \leq \lfloor \frac{n}{2} \rfloor$. The remaining task is to prove that the returned allocation with payment $(\mathbf{A}, \mathbf{p})$ is indeed envy-free.

Lemma 9. *At the moment right before Phase 2, no agent envies their out- and in-neighbours in G' if t_i is the payment to agent i.*

Lemma 10. *At the termination of Algorithm 3, the allocation with payment $(\mathbf{A}, \mathbf{p})$ is envy-free.*

Combining the above lemmas, Theorem 2 is proved.

4 Conclusion

In this paper, we study the problem of achieving envy-freeness with subsidy in the graph orientation model. For multigraphs, we prove that a subsidy of $\lfloor \frac{n}{2} \rfloor$ is necessary and sufficient in the worst case. Our proof is constructive; we design a polynomial-time algorithm that ensures the required subsidy does not exceed $\lfloor \frac{n}{2} \rfloor$. The bound cannot be improved even for simple graphs.

Our work also opens up several intriguing directions for future research. For example, in this paper, we assume additive valuations; exploring more general valuation functions would be a natural extension. As mentioned earlier, the tight bound for general monotone valuations is still unknown [Liu *et al.*, 2024], where the upper and lower bounds are $\frac{1}{2}n(n-1)$ and $n-1$ [Brustle *et al.*, 2020; Kawase *et al.*, 2024]. For monotone valuations with binary marginal values, it is proved in [Barman *et al.*, 2022] that $\Theta(n)$ is necessary and sufficient. It is interesting to see if $\Theta(n)$ is enough for orientation models. Additionally, beyond graphs, it would be valuable to extend this study to other (p, q)-bounded valuation settings, where each item may be relevant to more than two agents.

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