The Core of Approval-Based Committee Elections with Few Seats

Dominik Peters
CNRS, LAMSADE, Université Paris Dauphine - PSL
dominik.peters@lamsade.dauphine.fr

Abstract

In an approval-based committee election, the goal is to select a committee consisting of k out of m candidates, based on n voters who each approve an arbitrary number of the candidates. The *core* of such an election consists of all committees that satisfy a certain stability property which implies proportional representation. In particular, committees in the core cannot be "objected to" by a coalition of voters who is underrepresented. The notion of the core was proposed in 2016, but it has remained an open problem whether it is always non-empty. We prove that core committees always exist when $k \leqslant 8$, for any number of candidates m and any number of voters n, by showing that the Proportional Approval Voting (PAV) rule, proposed by Thiele in 1895, always satisfies the core when $k \leqslant 7$ and always selects at least one committee in the core when $k = 8$. We also develop an artificial rule based on recursive application of PAV, and use it to show that the core is non-empty whenever there are $m \leqslant 15$ candidates, for any committee size $k \leqslant m$ and any number of voters n. These results are obtained with the help of computer search using linear programs.

1 Introduction

The seminal work of Aziz *et al.* [2016; 2017] introduced a rigorous way of reasoning about voter representation in multi-winner elections. Their model considers *approval-based committee elections*, where the task is to identify a committee $W \subseteq C$ of k out of m candidates, based on the preferences of a set of voters N, with each $i \in N$ indicating a subset $A_i \subseteq C$ of candidates that i approves. Aziz *et al.* [2017] formulated a compelling axiom called Extended Justified Representation (EJR) which gives representation guarantees to every group of voters who approve sufficiently many candidates in common. Researchers discovered that voting rules developed 130 years ago by Thiele [1895] and Phragmén [1894] satisfy this or related axioms [Brill *et al.*, 2023; Aziz *et al.*, 2017; Janson, 2016]. Interesting *new* rules satisfying EJR have recently been developed [Aziz *et al.*, 2018; Peters and Skowron, 2020; Brill and Peters, 2023], with one of them (the "Method of Equal Shares") now in active use for the participatory budgets in several cities in Poland, Switzerland, and the Netherlands.

Aziz *et al.* [2016; 2017, Section 5.2] also defined another representation axiom that is significantly stronger than EJR, called *core stability* in analogy to a similar concept from cooperative game theory. A committee W is core stable if for every set $T \subseteq C$, there are not too many voters who prefer the set T to W, namely we have

$$\left|\left\{i \in N : |A_i \cap T| > |A_i \cap W|\right\}\right| < |T| \cdot \frac{n}{k}.$$

If this inequality were violated for some T, then the set of voters on the left-hand side could form a *blocking coalition* of a size that is large enough for the coalition to "deserve" to decide to include T in the committee.

The EJR property is weaker than core stability (as under EJR voters are only allowed to join the blocking coalition if they approve all the candidates in T, i.e., $|A_i \cap T| = |T|$), but in exchange there are several attractive voting rules satisfying EJR. On the other hand, Aziz *et al.* [2016; 2017] noted that "the core stability condition appears to be too demanding, as none of the voting rules considered in our work is guaranteed to produce a core stable outcome". No such voting rules have been discovered since. They conclude: "It remains an open question whether the core [is always] non-empty." This question remains open more than 8 years later.

Some amount of progress has been made, and in particular it is known that there always exist committees satisfying approximate variants of the core [Peters and Skowron, 2020; Jiang *et al.*, 2020; Munagala *et al.*, 2022b], and the core exists on single-peaked approval profiles [Pierczyński and Skowron, 2022] and on profiles where each candidate has at least k copies [Brill *et al.*, 2022].

To the best of my knowledge, the only known existence result that holds *in general* is that the core is non-empty for $k = 3$, which Cheng *et al.* [2020, Section 3.1] showed by case analysis. Cheng *et al.* [2020] conclude that a "major open question is the existence of deterministic stable committees in the Approval Set setting, generalizing our positive result for $k = 3$ to general k. We conjecture that such a stable committee always exists. Via computer-assisted search, we have shown that this conjecture holds for small numbers of voters and candidates ($m + n \leqslant 14$)."

It might seem surprising that the state of the art has not improved beyond these very small parameter values. In particular, there is a natural way of using mixed integer linear programming (ILPs) to search for counterexamples to core existence: fix m and k, and introduce a fractional variable for each possible ballot, indicating what fraction of the voters submit this ballot. Then, for every committee, enforce using binary variables that there is at least one successful core deviation. If an ILP solver determines that the resulting program is infeasible, this implies the non-emptiness of the core for m and k, for any number n of voters. Unfortunately, the size of this program grows rapidly, and they are not easy to solve even for very small sizes (Gurobi solves $m = 7$, $k = 5$, in 450s, but did not solve $m = 7$, $k = 4$ after 134 000s (37h) on 8 cores).[1]

In this paper, by deriving a new way of using solvers, we show that the core always exists for committee sizes up to $k = 8$, for any number of candidates (improving upon the previous result for $k = 3$). We also show that the core always exists when the number m of candidates is at most 15, for any $k \leqslant m$. Both results hold *for any number n of voters*.

These results are obtained by analyzing variants of *Proportional Approval Voting* (PAV), the voting rule proposed by Thiele [1895]. This voting rule works by maximizing a carefully chosen objective function over the set of all committees of size k. We show via linear programs that PAV always selects a core-stable committee when $k \leqslant 7$, and that it always selects at least one core-stable committee when $k = 8$, perhaps tied with other committees that fail core-stability. This performance of the PAV rule is remarkable, and contrasts with other rules such as the Phragmén rule and the Method of Equal Shares which fail the core for $k = 6$ and $k = 7$, respectively (see Footnote 3). In many applications, the number of seats to fill will be 8 or fewer, and thus our result suggests that PAV is a good rule for such contexts.[2]

For the results about limited numbers of candidates, we consider a recursive version of PAV where, if the PAV committee fails the core because some set T has too much support, we then re-compute PAV subject to the constraint that $T \subseteq W$ and without taking into account the voters who were part of the blocking coalition. If the result still fails core-stability, we add additional constraints. We show that if $m \leqslant 15$, this process always terminates with a core-stable committee. However, there are examples where the rule is not in the core for $m = 16$.

As mentioned, these results were obtained with the help of linear programming. This becomes feasible even for these quite large sizes because we can fix *one* committee, and add constraints that this committee is the one selected by the voting rule under consideration. This is much simpler than a program that needs constraints for all possible committees. Linear programming has been used before to analyze sequential versions of PAV [Skowron, 2021; Sánchez-Fernández *et al.*, 2017]. The infeasibility of the relevant programs can be compactly certified via Farkas' lemma, allowing efficient verification of our results without having to trust a solver. Code for these tasks is available at https://github.com/DominikPeters/core-few-candidates/. Some details and proofs have been omitted in this conference version due to space constraints; these can be found in the full version at arXiv:2501.18304 [Peters, 2025].

[1] The same problem can also be encoded as an SMT problem on linear arithmetic. This can sometimes lead to faster solve times, but it is also only feasible for very small sizes.

[2] Note, however, that even for $k = 6$, PAV may select committees that fail laminar proportionality [Peters and Skowron, 2020] and are intuitively unfair. A three-voter example for $k = 6$ is $A_1 = \{c_1, c_2, c_3\}$, $A_2 = \{c_1, c_2, c_4\}$, and $A_3 = \{c_5, c_6, c_7\}$ where $W = \{c_1, c_2, c_3, c_5, c_6, c_7\}$ is a global PAV committee (tied with others) while laminar proportionality demands that $\{c_1, c_2, c_3, c_4\} \subseteq W$.

2 Related Work

Barriers to core existence. Proving that the core is nonempty is difficult because several natural strategies are known not to work. Importantly, all known voting rules that satisfy weakenings of the core such as EJR fail the core, including the PAV rule [Aziz *et al.* 2017, Example 6; Peters and Skowron 2020, Section 1]. Peters and Skowron [2020, Theorem 10] show that every welfarist rule (one that depends only on voter utilities) must fail the core. They also show that every voting rule satisfying the Pigou–Dalton principle (which says that outcomes that induce a more equitable social welfare distribution should be preferred; this is satisfied by PAV) cannot satisfy the core, and indeed cannot provide better than a 2-approximation to it Peters and Skowron [2020, Theorem 5].

Computational complexity. It is NP-hard to compute the PAV rule [Aziz *et al.*, 2015, Corollary 1], but it is fixed-parameter tractable for a variety of parameters [Yang and Wang, 2023]. A local search variant of PAV retains its proportionality properties and can be computed in polynomial time [Aziz *et al.*, 2018] for an appropriately chosen tolerance parameter [Kraiczy and Elkind, 2024]. The problem of checking whether a given committee is in the core is coNP-complete [Brill *et al.*, 2022, Theorem 5.3] and remains hard even when every voter approves at most 6 candidates, and each candidate is approved by at most 2 voters [Munagala *et al.*, 2022a, Theorem 1]. The verification problem is also hard to approximate to within a factor better than $1 + 1/e$ [Munagala *et al.*, 2022a, Theorem 2], though a logarithmic approximation algorithm exists [Munagala *et al.*, 2022a, Theorem 3].

More general settings. We work in the model of approval-based committee elections. If non-approval preferences are allowed (such as cardinal additive valuations), the core may be empty [Fain *et al.* 2018, Appendix C; Peters *et al.* 2021, Example 2]. For participatory budgeting applications, one can replace the cardinality constraint in the definition of a committee by a knapsack constraint. For this non-unit cost setting, approval votes have several interpretations [Rey and Maly, 2023, Section 3.4.2]. One option is *cost utilities*, which measures a voter's utility by the total cost of approved winning projects. For this utility model, the core may be empty [Maly, 2023]. For *cardinality utilities*, where the voter's utility is the number of approved winning projects, the non-emptiness of the core is an open question.

Approximate core. Core stability can be relaxed through multiplicative approximations, in two main ways. Say that a committee W is in the (α, β)-core, $\alpha, \beta \geqslant 1$, if we have $|\{i \in N : |A_i \cap T| > \alpha \cdot |A_i \cap W|\}| < \beta \cdot |T| \cdot \frac{n}{k}$ for every potential deviation T. For $\alpha = \beta = 1$, this is the core; for $\alpha > 1$, every member of the blocking coalition must increase their utility by a factor of α; for $\beta > 1$, we require that blocking coalitions must be larger than usual by a factor of β.

Peters and Skowron [2020] show that PAV is in the $(2, 1)$-core, and that the Method of Equal Shares is in the $(\log k, 1)$-core (or a mild relaxation of that concept). Munagala *et al.* [2022b] show that the $(9.27, 1)$-core is non-empty even for general additive valuations. Fain *et al.* [2018, Appendix C] show that the $(1 + \varepsilon, 1)$-core is non-empty if it is additionally additively relaxed. Jiang *et al.* [2020] show that the $(1, 16)$-core is non-empty, via rounding a stable lottery [Cheng *et al.*, 2020]; they conjecture that at least the $(1, 2)$-core is always non-empty. A similar rounding technique has been used to analyze Condorcet winning sets [Charikar *et al.*, 2024].

Fractional models. Analogs of core-stability have also been defined in fractional models. For example, Aziz *et al.* [2020] consider "fair mixing" in an approval-based model, where the output is a probability distribution over candidates (which can be interpreted as a division of a budget). They show that the rule maximizing Nash welfare (which is related to the PAV rule) satisfies core-stability. Fain *et al.* [2016] obtain the same result in a more general model with additive linear utilities. They also show that a core-stable outcome exists for fractional committees (which can be viewed as a probability distribution where each candidate receives mass at most $1/k$) via Lindahl equilibrium which is known to exist from fixed-point theorems [Foley, 1970; Munagala *et al.*, 2022b] and via convex programming [Kroer and Peters, 2025]. Various weakenings of the core have also been studied in approval-based fair mixing and related models [see, e.g., Brandl *et al.* 2021; Suzuki and Vollen 2024; Bei *et al.* 2025].

3 Preliminaries

Let $C = \{c_1, \ldots, c_m\}$ be a set of m candidates. Let $k \in \{1, \ldots, m\}$ be the committee size. A *committee* is a subset $W \subseteq C$ of winning candidates with $|W| = k$.

Let $\mathcal{A}$ denote the set of non-empty subsets of C, which we will refer to as *approval sets* or *ballots*. A *profile* is a map $P : \mathcal{A} \to \mathbb{Q}_{\geqslant 0}$ with $\sum_{A \in \mathcal{A}} P(A) = 1$, where $P(A)$ indicates the fraction of the voters that approve exactly the candidates in A. Given an approval set $A \in \mathcal{A}$ and a committee W, the *utility* of the committee for A is the number of approved committee members: $u_A(W) = |A \cap W|$.

A nonempty set $T \subseteq C$ with $|T| \leqslant k$ is called a *potential deviation*. Given a profile P, a committee W is *core stable* if for every potential deviation T, we have

$$\sum_{A \in \mathcal{A}: u_A(T) > u_A(W)} P(A) < \frac{|T|}{k}.$$

Otherwise, T is called a *deviation* (or a *successful deviation*) from W. The *core* is the set of committees that are core stable.

The nth *harmonic number* is $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. The *PAV score* assigned to a committee W by ballot $A \in \mathcal{A}$ is

$$\text{PAV-score}_A(W) = H(u_A(W)).$$

Given a profile P, the *PAV score* of W under profile P is

$$\text{PAV-score}_P(W) = \sum_{A \in \mathcal{A}} P(A) \cdot \text{PAV-score}_A(W).$$

A committee is a *global PAV committee* if it has the highest PAV score. The *PAV rule* selects all global PAV committees.

We will also be interested in local maxima of the PAV objective, i.e., committees that have a weakly higher PAV score than any committee obtained by performing a single swap. For a committee W, $x \in W$ and $y \in C \setminus W$, we write $W_{xy} = W \setminus \{x\} \cup \{y\}$ for the committee obtained by replacing x with y. Given a profile P and a fixed committee W, we write

$$\Delta_{P,x,y} = \text{PAV-score}_P(W_{xy}) - \text{PAV-score}_P(W)$$

for the increase in PAV score resulting from this swap. For a ballot A, we define $\Delta_{A,x,y}$ analogously. Then we say that W is a *local PAV committee* if $\Delta_{P,x,y} \leqslant 0$ for all x and y. Note that every global PAV committee is also a local PAV committee.

The following lemma shows that a local PAV committee never admits deviations of certain kinds: disjoint deviations and deviations that contain only a single unelected candidate.

Lemma 3.1. *Let W be a local PAV committee, and let T be a potential deviation. Suppose we have*

$$\text{(i) } T \cap W = \emptyset, \quad \text{or} \quad \text{(ii) } |T \setminus W| \leqslant 1.$$

Then T is not *a deviation from W.*

Proof. (i) is proved by Brill *et al.* [2022, Theorem 3.2 and Remark 3.4] using a swapping argument.

(ii) If $T \setminus W = \emptyset$, then $T \subseteq W$, and there are no approval ballots that strictly prefer T to W. If $T \setminus W = \{c\}$, then for every $A \in \mathcal{A}$ with $u_A(T) > u_A(W)$, we have $c \in A$. Writing $\ell = |T|$, if T were a deviation, we would thus have a set S of ballots forming ℓ/k of the profile, who all approve $c \notin W$, and who all have utility $u_A(W) < u_A(T) \leqslant \ell$. Thus, we have a violation of the EJR+ axiom of Brill and Peters [2023] which local PAV committees are known to satisfy. □

Note that Theorem 3.1(ii) implies that PAV satisfies the core when $k = m - 1$ or $k = m$.

We recall the following well-known result about systems of linear inequalities, providing a certificate of infeasibility.

Lemma 3.2 (Farkas' Lemma)**.** *Let $A \in \mathbb{Q}^{mn}$ be an $m \times n$ matrix, and let $b \in \mathbb{Q}^m$. Then the following are equivalent.*

(i) There does not exist $x \in \mathbb{Q}^n$ with $Ax \leqslant b$.

(ii) There exists an integer vector $y \in \mathbb{Z}^m_{\geqslant 0}$ such that $y^T b < 0$ and $A^T y \geqslant 0$.

Thus, by exhibiting the integer vector y, one can prove the infeasibility of the system $Ax \leqslant b$.

4 Small Committee Size

In this section, we discuss core-stable committees when the committee size k is small. We give existence results when $k \leqslant 8$, separately handling the cases $k \leqslant 7$ and $k = 8$.

4.1 Committee Size $k \leqslant 7$

For committee sizes up to 7, core-stable committees always exist because every local PAV committee is in the core. This establishes existence of core-stable committees, but also indicates that PAV rule is a very good rule for smaller committee sizes.[3] Like other proofs that PAV is proportional, our proof reasons about the change in PAV score caused by certain swaps. In particular, it shows that if W is a committee with $|W| \leqslant 7$ such that T is a successful deviation, then on average, replacing an element of $W \setminus T$ by an element of $T \setminus W$ will increase the PAV score of W. Hence there exists at least one such swap that increases the PAV score of W, and hence it cannot be a local PAV committee. The proof is omitted due to space constraints, but can be found in the full version [Peters, 2025]. To establish a key inequality, the proof refers to the result of a computer enumeration of all possible ballot types, which can be reproduced using code available on GitHub.

Theorem 4.1. *When $k \leqslant 7$, every local PAV committee is in the core.*

The result of Theorem 4.1 can also be obtained by linear programming. Suppose that the result is false, so that there exists a profile P and a local PAV committee W such that some T is a successful deviation. Without loss of generality, there exists such an example with $W = \{c_1, \dots, c_k\}$. Note that P then forms a solution to the following system of linear inequalities, where we may assume that $C = W \cup T$:[4]

$$\begin{aligned} \sum_{A \in \mathcal{A}} P(A) &= 1 \\ \Delta_{P,x,y} &\leqslant 0 \qquad \text{for all } x \in W \text{ and } y \in C \setminus W \\ \sum_{A \in \mathcal{A}: u_A(T) > u_A(W)} P(A) &\geqslant \frac{|T|}{k} \\ P(A) &\geqslant 0 \qquad \text{for all } A \in \mathcal{A} \end{aligned} \tag{1}$$

Thus, Theorem 4.1 is proven if the system (1) does *not* have a feasible solution, for all potential deviations T. This can be certified using Farkas' lemma, and the proof of Theorem 4.1 implicitly constructs such a certificate solution for every T.

Due to the following remark, it is possible to compute a core-stable outcome in $O(m^2 n)$ time whenever $k \leqslant 7$.

Remark 4.2. *Let $k \leqslant 7$ and $\varepsilon = 0.1/k^2$. By solving linear programs* [code on GitHub]*, one can check that every committee that is ε-local-swap-stable (i.e., $\Delta_{P,x,y} \leqslant \varepsilon$ for all x and y) is in the core. An ε-local-swap-stable and thus a core-stable committee can be found by performing $O(k^2 \ln k) = O(1)$ many ε-improving swaps [see Aziz* et al. *2018, Proposition 1], each of which takes $O(m^2 n)$ time to find. It turns out that for $\varepsilon = 1/k^2$, ε-local-swap-stable committees need not be in the core, even though this value of ε is enough to ensure that the committee satisfies EJR [Aziz* et al.*, 2018, Theorem 1]. For example, in the profile with $P(\{a, b\}) = P(\{a, c\}) = 0.25$ and $P(\{d, e, f, g, h\}) = 0.5$, for $k = 6$, the committee $\{a, d, e, f, g, h\}$ fails the core due to $T = \{a, b, c\}$, but it is $1/40$-local-swap-stable, and $1/40 < 1/36 = 1/k^2$.*

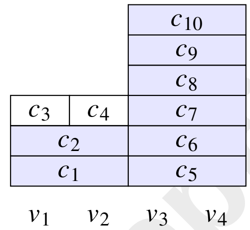
[3] There are examples where the sequential Phragmén rule fails core (and even EJR) for $k = 6$, and where MES fails core for $k = 7$. These counterexamples work even for the party-approval setting [Brill *et al.*, 2022], where each candidate can be placed in the committee several times. For Phragmén, take the 3-voter profile (ab, bc, ac), where Phragmén can elect $ababab$, with $T = \{c, c, c, c\}$ forming a deviation. For MES, take the 7-voter profile $(ab, ac, ad, bcd, bcd, bcd, bcd)$, where $bbbbbaa$ is an outcome of MES, with $T = \{c, c, c, d, d, d\}$ forming a deviation. (In these examples, there are other tied outcomes in the core, and I don't know if unique examples exist.)

[4] We may make this assumption because a counterexample on a larger C remains a counterexample when restricted to $W \cup T$, because a local PAV committee remains a local PAV committee after deleting candidates outside the committee.

4.2 Committee Size $k = 8$

Theorem 4.1 does not hold for $k = 8$: There are profiles where some local (and even global) PAV committee is not in the core.

Example 4.3 (PAV may fail core for $k = 8$)**.** *Consider an instance with 4 voters, v_1 approving $\{c_1, c_2, c_3\}$, and v_2 approving $\{c_1, c_2, c_4\}$, and the other 2 voters approving $\{c_5, c_6, c_7, c_8, c_9, c_{10}\}$. This profile is depicted below, where each voter approves the candidates above the voter's label.*



On this profile, $W = \{c_1, c_2, c_5, c_6, c_7, c_8, c_9, c_{10}\}$ is a global PAV committee (indicated in blue in the picture). However, W is not in the core: consider $T = \{c_1, c_2, c_3, c_4\}$, which has support from $\frac{1}{2}$ of the voters, and $|T|/k = \frac{1}{2}$. □

Note, however, that in Theorem 4.3, there is more than one global PAV committee. In particular, $W' = \{c_1, c_2, c_3, c_5, c_6, c_7, c_8, c_9\}$ (obtained by removing c_{10} and adding c_3) is also a global PAV committee and it is in the core.

It turns out that Theorem 4.3 is essentially the only example where a PAV committee fails to be core-stable for $k = 8$, as all such examples share the same structure.

Lemma 4.4. *Let P be a profile and suppose that W with $|W| = 8$ is a local PAV committee that is not in the core due to objection T. Then there exist distinct $a, b \in W$ and distinct $x, y \in C \setminus W$ such that $T = \{a, b, x, y\}$. In addition,*

(i) one quarter of the voters submit ballots A such that $A \cap (W \cup T) = \{a, b, x\}$ and another quarter submit ballots with $A \cap (W \cup T) = \{a, b, y\}$,

(ii) the remaining half of the voters submit ballots that are disjoint from T, and

(iii) the PAV score of W is reduced by exactly $1/12$ if any one member of $W \setminus \{a, b\}$ is removed.

Proof. We first check that if W is not in the core, then any core objection must use a T with $|T| = 4$ and $|W \cap T| = 2$. This can be deduced using the linear programming approach behind Theorem 4.1; by iterating through all possible T, we find that the system (1) has a solution only for T satisfying the condition in the theorem statement.

Now fix such a $T = \{a, b, x, y\}$. Assume that there exists a profile P where W is a local PAV committee with successful deviation T but that violates any of the conditions (i)–(iii). Then if we delete all candidates outside $W \cup T$ from the profile, it would still fail (i)–(iii). Thus, for purposes of making the following linear programs finite, we may assume that $C = W \cup T$ (so $|C| = 10$).

To prove (i), we solve the following four linear programs:

$$\begin{aligned}
\text{maximize } P(\{a,b,x\}) &\text{ subject to } P \text{ satisfying (1)}\\
\text{minimize } P(\{a,b,x\}) &\text{ subject to } P \text{ satisfying (1)}\\
\text{maximize } P(\{a,b,y\}) &\text{ subject to } P \text{ satisfying (1)}\\
\text{minimize } P(\{a,b,y\}) &\text{ subject to } P \text{ satisfying (1)}
\end{aligned}$$

The optimal solutions to all these programs is $\frac{1}{4}$.

To prove (ii), iterate through all ballots $A \in \mathcal{A}$ with $A \cap T \neq \emptyset$, except for $\{a, b, x\}$ and $\{a, b, y\}$. For each of these ballots, solve the following linear program:

$$\text{maximize } P(A) \text{ subject to } P \text{ satisfying (1)}$$

For each such A, the optimal solution of the program is 0. To prove (iii), iterate through all $c \in W \setminus \{a, b\}$ and solve the following programs, both again subject to P satisfying (1):

$$\begin{aligned}
&\text{maximize PAV-score}_P(W \setminus \{c\}) - \text{PAV-score}_P(W)\\
&\text{minimize PAV-score}_P(W \setminus \{c\}) - \text{PAV-score}_P(W)
\end{aligned}$$

The optimal solutions to these two programs are $-1/12$. □

The claims made in this proof about the optimal values of the various linear programs can be certified by exhibiting solutions to the dual programs. These certificates (using exact fractions, not floating point numbers) are available on GitHub, together with a script checking their validity without a solver.

As we discussed, Theorem 4.3 shows an example of a global PAV committee that is not core-stable, but there are other global PAV committees for the same profile that are core-stable. Thanks to Theorem 4.4, we deduce that the same holds for *all* counterexamples. Hence, for every instance, at least one global PAV committee is in the core, and thus the core is always non-empty for $k = 8$.

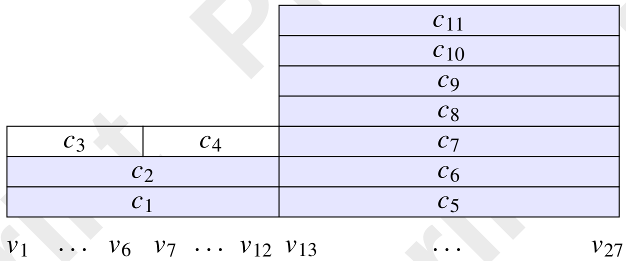
Theorem 4.5. *When $k = 8$, some global PAV committee is in the core.*

Proof. Let $k = 8$ and let P be a profile. If on P, all global PAV committees are in the core, we are done. So suppose that W is a global PAV committee that is not core stable due to objection T. From Theorem 4.4, there exist distinct $a, b \in W$ and distinct $x, y \in C \setminus W$ such that $T = \{a, b, x, y\}$. Take any $c \in W \setminus \{a, b\}$. Then the committee $W' = W \setminus \{c\} \cup \{x\}$ has the same PAV score as W, because the removal of c causes a decrease in PAV score of $1/12$ and the addition of x causes an increase of at least $\frac{1}{4} \cdot \frac{1}{3} = 1/12$ due to the quarter of voters from (i) with ballots A such that $A \cap (W \cup T) = \{a, b, x\}$. Thus, W' is also a global PAV committee. We now show that W' is in the core.

If not, we can apply Theorem 4.4 to W' which gives us an objection $T' = \{a', b', x', y'\}$ to W'. Clearly, voters with ballots such that $A \cap (W \cup T) = \{a, b, x\}$ are not part of a blocking coalition because $\{a, b, x\} \subseteq W'$. Thus, we deduce that $a, b \notin T$ from (ii). Thus, the voters with ballots such that $A \cap (W \cup T) = \{a, b, y\}$ are also not supporters of T'. Then from part (i) we deduce that the only members of W that are approved by any voters in P are a, b, a', and b'. Thus, there exists a member of $W \setminus \{a, b\}$ who is not approved by any voter, so the removal of that member does not lead to a reduction in PAV score, contradicting (iii). □

4.3 Committee Size $k \geqslant 9$

The PAV-based technique that worked for up to $k = 8$ does not continue to work for $k = 9$, since there are examples where there is a unique global PAV committee which fails to be in the core. The following example has this property, and is minimal with respect to the number of voters ($n = 27$).



Aziz *et al.* [2017, Ex. 6] gave an example where PAV uniquely selects a non-core-stable committee for $k = 10$ and $n = 20$.

5 Few Candidates

The goal of this section is to show that there always exists a core-stable committee on instances with $m \leqslant 15$ candidates. From the results in Section 4, this is clearly true when $k \leqslant 8$. By Theorem 3.1(ii), this is also true when $k = m - 1$ or $k = m$. But it is not clear when $k \in \{9, \dots, m - 2\}$.

Inspecting the examples in Section 4 where PAV fails the core, we see that they are well-structured. Indeed, they are even *laminar instances* in the sense of Peters and Skowron [2020, Definition 2], and it is easy to see that on these profiles, a core-stable committee does exist. Thus, there is some hope to prove existence of core-stable committees by "patching" the PAV committee when it fails to be in the core.

We will define an artificial rule, based on PAV, and we will show that it satisfies core stability for up to $m = 15$ candidates. We call it the *recursive PAV rule*. On a high level, the rule first computes a local PAV committee, and checks if it satisfies the core. If so, it returns it. If not, and T is a deviation from W, it then deletes all voters who prefer T to W, and computes a local PAV committee with respect to the remaining voters, but subject to the constraint that $T \subseteq W$. It then checks if the result is in the core; if not, it adds additional constraints until it reaches a core-stable committee. This rule is formally described using pseudocode in Algorithm 1.

For example, in the profile of Theorem 4.3, PAV selects the committee W indicated there in blue (among other tied committees). This committee is blocked by $T = \{c_1, c_2, c_3, c_4\}$. Thus, the recursive PAV rule would now fix all members of T as winners, and maximize the PAV-score with respect to voters v_3 and v_4, obtaining the committee $W^* = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ (among other tied committees). This committee is in the core, so the rule terminates. Note that W^* is not itself selected by PAV.

Algorithm 1 Recursive PAV rule

Input: A profile P and a committee size k
Output: A committee W
$\mathcal{A}' \leftarrow \mathcal{A}$, set of active ballots
$F \leftarrow \emptyset$, set of *fixed* candidates
while true **do**
  If $|F| > k$, the algorithm **fails**
  $W \leftarrow$ any committee locally maximizing the PAV score w.r.t. the ballots in $\mathcal{A}'$ and subject to $F \subseteq W$
  if there exists a successful deviation T from W **then**
    $F \leftarrow F \cup T$
    $\mathcal{A}' \leftarrow \mathcal{A}' \setminus \{A \in \mathcal{A} : u_A(T) > u_A(W)\}$
  else
    return W

This method is reminiscent of the Greedy Cohesive Rule [Peters *et al.*, 2021], which similarly repeatedly patches a committee until it satisfies the representation axiom FJR.

5.1 Analysis of the Method

Fix a number of candidates m and a committee size k.

A list $(W_1, T_1), (W_2, T_2), \dots, (W_r, T_r)$ is called a *potential history* if for each $t \in [r]$, we have that W_t is a committee, T_t is a potential deviation, and $T_1 \cup \dots \cup T_{t-1} \subseteq W_t$.

Definition 5.1. *A potential history $(W_1, T_1), \dots, (W_r, T_r)$ is a* history *if there exists a profile P such that for each $t \in [r]$ we have that T_t is a* successful *deviation from W_t, and that for all $x \in W \setminus (T_1 \cup \dots \cup T_{t-1})$ and $y \in C \setminus W$, we have*

$$\sum_{A \in \mathcal{A}_t} P(A) \cdot \text{PAV-score}_A(W_t) \geqslant \sum_{A \in \mathcal{A}_t} P(A) \cdot \text{PAV-score}_A(W_{xy})$$

where $\mathcal{A}_t = \{A \in \mathcal{A} : u_A(T_s) \leqslant u_A(W_s) \text{ for } s \in \{1, \dots, t-1\}\}$ is the set of "active" ballots. That is, W_t locally maximizes the PAV score among all committees that include all prior deviations, taking only those voters into account that did not participate in prior deviations.

Thus, a history provides a trace of the execution of Algorithm 1 for some profile. The following result states that it is enough to analyze the set of histories to determine if Algorithm 1 always terminates with a core-stable committee.

Proposition 5.2. *Suppose that for every history $(W_1, T_1), \dots, (W_r, T_r)$, we have $|T_1| + \dots + |T_r| \leqslant k$. Then a core-stable committee always exists for m and k.*

Proof. Let P be a profile, and run Algorithm 1 on it. By the assumption, in each iteration, $|F| \leqslant |T_1| + \dots + |T_r| \leqslant k$, so the algorithm does not fail. By the if-clause, if the algorithm terminates, it returns a committee that is core-stable. Thus, it suffices to show that the algorithm terminates.

Note that after each iteration of the algorithm, it either terminates or it has found a successful deviation. Suppose iteration r has ended without the algorithm terminating. The sequence of committees and deviations $(W_1, T_1), \dots, (W_r, T_r)$ identified by the algorithm up to iteration r forms a history. Since $|T_t| \geqslant 1$ for all t, it follows from $|T_1| + \dots + |T_r| \leqslant k$ that $r \leqslant k$. So it must terminate after at most $k + 1$ iterations. □

Algorithm 2 Finding all histories

Input: Number m of candidates and a committee size k
Output: A collection of all histories and Farkas certificates
$\mathcal{H}_0 \leftarrow \{\emptyset\}$, the empty history
for $t = 1, 2, \dots$ **do**
  for all $H \in \mathcal{H}_{t-1}$ **do**
    for all potential continuations (W_t, T_t) **do**
      if (W_t, T_t) is not canonical w.r.t. H **then**
        continue
      Set $H' \leftarrow H + (W_t, T_t)$
      Solve LP to check if H' is a history
      If yes, add H' to $\mathcal{H}_t$
      If no, generate a Farkas certificate
  if $\mathcal{H}_t = \emptyset$ **then**
    break

Thus, to prove the existence of core-stable committees, it suffices to enumerate all histories and check that they fulfil the condition of Theorem 5.2. Given a potential history, one can check using an LP solver whether it is a history by checking whether the system of linear inequalities in Theorem 5.1 has a solution. This way, we can compute the set of histories using a standard breadth-first search, as shown in Algorithm 2. A key insight to speed up the search is that we may break symmetries and only consider "canonical" histories in our enumeration. For example, we may assume without loss of generality that W_1, the first committee of the history, is $\{c_1, \dots, c_k\}$. Similarly, we do not need to consider all potential deviations T_1: given our choice of W_1, the candidates in W_1 are indistinguishable to each other, as are the candidates in $C \setminus W_1$. Thus, it suffices to take one deviation for each possible combination of the sizes of $|T_1 \cap W_1|$ and of $|T_1 \cap (C \setminus W_1)|$. Similar symmetry-breaking conditions apply for later steps.

For example, for $m = 15$ and $k = 13$, Algorithm 2 produces the following set of (canonical) histories, where we write $W_1 = \{c_1, \dots, c_{13}\}$ and $W_2 = \{c_1, \dots, c_{11}, c_{14}, c_{15}\}$.

$\emptyset$, the empty history
$(W_1, \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_2, c_3, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_2, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_2, c_3, c_4, c_5, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_2, c_3, c_4, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_2, c_3, c_4, c_5, c_6, c_{14}, c_{15}\})$
$(W_1, \{c_1, c_{14}, c_{15}\}), \quad (W_2, \{c_2, c_{12}, c_{13}\})$
$(W_1, \{c_1, c_{14}, c_{15}\}), \quad (W_2, \{c_2, c_3, c_{12}, c_{13}\})$
$(W_1, \{c_1, c_2, c_3, c_{14}, c_{15}\}), \quad (W_2, \{c_4, c_5, c_{12}, c_{13}\})$
$(W_1, \{c_1, c_2, c_{14}, c_{15}\}), \quad (W_2, \{c_3, c_{12}, c_{13}\})$
$(W_1, \{c_1, c_2, c_{14}, c_{15}\}), \quad (W_2, \{c_3, c_4, c_{12}, c_{13}\})$
$(W_1, \{c_1, c_2, c_{14}, c_{15}\}), \quad (W_2, \{c_3, c_4, c_5, c_{12}, c_{13}\})$

By running Algorithm 2 for $m = 15$ and $k = 9, \dots, 13$, we obtain the following result. (Note that existence for $m = 15$ implies existence for all $m \leqslant 15$.)

Theorem 5.3. *If $m \leqslant 15$, a core-stable committee exists.*

$k =$	9	10	11	12	13
# of canonical histories	7	11	15	20	15
# of Farkas witnesses	20 476	25 313	18 567	43 140	6 877
time (s) to check Farkas	2 648	3 301	2 087	5 857	725

Table 1: Statistics about the histories for $m = 15$.

The computations establishing Theorem 5.3 can be verified based on Farkas certificates: the code repository includes, for each history and each possible extension of the history that induces an infeasible system of linear inequalities, a Farkas witness. Each witness is a list of about $t \cdot k \cdot (m - k)$ integers, where t is the length of the history, corresponding to the constraints in Theorem 5.1, and verifying the correctness of the witness requires checking about 2^m inequalities. In total, there are 114 373 witnesses (taking 125 MB). Verifying their validity using a simple script [available on GitHub] that performs exact fractional computations (without calling a solver) takes about 4 hours on 8 cores (see Table 1).

The recursive PAV rule fails for $m = 16$, $k \in \{10, 11\}$. For $m = 16$, $k = 10$, the smallest failure example I have found has 40 448 550 voters, though the ILP for minimizing this number did not converge within a reasonable amount of time. The example is available online.[5] The recursive PAV rule *does* work for $m = 16$, $k \in \{9, 12, 13, 14\}$, and it is plausible that it can be fixed *ad hoc* for $k \in \{10, 11\}$, so it is likely that the core continues to exist for $m = 16$.

6 Droop Quota

Our definition of core stability is based on the intuition that a $1/k$ fraction of the voters is "entitled" to decide on one of the committee members, and that an ℓ/k fraction is entitled to decide on ℓ committee members. The quantity $1/k$ is known as the *Hare quota*. But one can also define core stability based on the *Droop quota*, according to which each group of voters that makes up a strictly larger fraction than $1/(k+1)$ is entitled to decide on one committee member. Thus, a committee W is *Droop core stable* if for every potential deviation T, we have

$$\sum_{A \in \mathcal{A} : u_A(T) > u_A(W)} P(A) \leqslant \frac{|T|}{k+1}.$$

This is a stricter condition than the normal core, so if W is Droop core stable then it is also core stable.

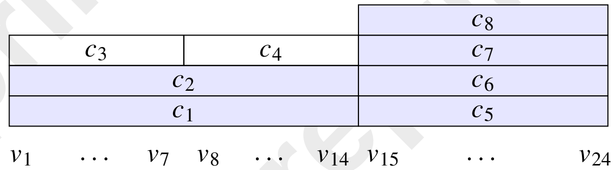
For most proportionality notions considered in the literature on approval-based committee elections, passing to the more demanding Droop quota does not cause many issues. For example, PAV still satisfies EJR when defined with the Droop quota, and analogous statements are true for many pairs of voting rules and representation axioms [Janson, 2018].[6]

[5]The example witnessing this failure induces the history
$(\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}, \{c_0, c_{10}, c_{11}\})$,
$(\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}\}, \{c_{13}, c_{14}, c_{15}\})$,
$(\{c_0, c_1, c_2, c_3, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}\}, \{c_4, c_5, c_6, c_7, c_8\})$.

[6]However, regarding strategic aspects, impossibility theorems become somewhat more expansive when passing to the Droop quota [Peters, 2018, Section 5.3].

Unfortunately, our positive results do not extend to the Droop core. While PAV satisfies the core for up to $k = 8$ (Section 4), it violates the Droop core already for $k = 6$.

Example 6.1 (PAV may fail the Droop core for $k = 6$)**.** *Consider the instance depicted below:*



On this profile, $W = \{c_1, c_2, c_5, c_6, c_7, c_8\}$ is the unique global (and unique local) PAV committee for $k = 6$. However W is not in the Droop core: consider $T = \{c_1, c_2, c_3, c_4\}$, which has support from $\frac{14}{24} \approx 0.583$ of the voters, while $|T|/(k+1) = \frac{4}{7} \approx 0.571$ is strictly smaller. □

This example is minimal, so the Droop core is non-empty when $k \leqslant 5$. Running the recursive PAV rule (Algorithm 1) with the Droop quota stops working even for $m = 10$, $k = 6$.

7 Conclusions

Based on the computations of this paper, we know that the core is non-empty for all small instances. This should probably strengthen our belief that the core is always non-empty. However, the recursive PAV method we defined to establish the result stops working for 16 or more candidates, so it seems doubtful that analyzing this method would allow proving a general existence result. Conversely, finding a counterexample to core existence will also be challenging since it will need to be large. For the Droop quota, however, it even remains unknown whether core always exists for $k = 6$ and $m = 10$.

Our approach was based on linear programming and allowed us to reason independently of the number of voters. The PAV rule and its variants are well-suited for these LP formulations. Finding core counterexamples for many other rules is not possible using similar linear programs. For example, the Method of Equal Shares (MES) [Peters and Skowron, 2020] or the sequential Phragmén method [Phragmén, 1894; Janson, 2016] do not admit the same kind of linear formulations (one would need to multiply variables corresponding to ballot frequencies with variables corresponding to ρ-values or to loads). The lack of such a linear formulation can be formally established using the techniques of Xia [2025]. In part because computer search is difficult for these rules, to the best of my knowledge, there is no known profile where both PAV and MES fail core-stability simultaneously, or a profile where the rule that maximizes the PAV score among all *priceable* committees [Peters and Skowron, 2020] fails core-stability.

Maly [2023] presents an example in the participatory budgeting setting with cost utilities where the core is empty. That example uses only 3 voters. It would be interesting to see if computer-aided methods could establish that for committee elections, the core is always non-empty for $n = 3$ voters. Note that in this case, candidates can be specified via the set of voters that approve the candidate, so there are only 2^3 different types of candidates, and thus a profile can be specified via variables that indicate how many candidates of each type exist.

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