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Generalized Safe Conditional Syntax Splitting of Belief Bases

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Abstract

Splitting techniques in knowledge representation help focus on relevant parts of a belief base and reduce the complexity of reasoning generally. In this paper, we propose a generalization of safe conditional syntax splittings that broadens the applicability of splitting postulates for inductive inference from belief bases. In contrast to safe conditional syntax splitting, our generalized notion supports syntax splittings of a belief base Δ where the subbases of Δ may share atoms and nontrivial conditionals. We illustrate how this new notion overcomes limitations of previous splitting concepts, and we identify genuine splittings, separating them from simple splittings that do not provide benefits for inductive inference from Δ. We introduce adjusted inference postulates based on our generalization of conditional syntax splitting. We evaluate several inductive inference operators with respect to these postulates, and show that generalized safe conditional syntax splitting is a strictly stronger requirement for inductive inference operators, covering more syntax splitting applications.

1 Introduction

For epistemic reasoning, both from a cognitive point of view and also from the point of view of effective implementations, it is often vital to focus on the relevant parts, and leaving aside facts and knowledge irrelevant for the question at hand, thus enabling local reasoning [Pearl, 1988]. This is the basic motivation underlying the concept of syntax splitting [Parikh, 1999; Peppas *et al.*, 2015; Kern-Isberner and Brewka, 2017], and of the related idea of minimum irrelevance [Weydert, 1998]. Under the motto "syntax splitting = relevance + independence", respecting syntax splitting was formalized for inductive inference from conditional belief bases [Kern-Isberner *et al.*, 2020], taking splittings over a belief base Δ into account where the subbases Δ_1, Δ_2 are given over disjoint subsignatures of Δ. This condition is a severe restriction in practice because full disjointness is often not the case. The concept of conditional syntax splitting [Heyninck *et al.*, 2023] is an approach to overcome this restriction by allowing Δ_1 and Δ_2 to overlap syntactically. A safety condition ensures that semantic (conditional) independence holds given the joint atoms, enabling local reasoning within the subbases. Furthermore, the postulate of conditional independence (CInd) for safe conditional splittings precisely characterizes avoiding the drowning effect [Pearl, 1990; Benferhat *et al.*, 1993], yielding the first formal definition of the notorious drowning problem that had been described before only by specific examples [Heyninck *et al.*, 2023]. However, it has been shown recently that the safety condition in [Heyninck *et al.*, 2023] has the undesirable consequence that every conditional in the intersection of Δ_1 and Δ_2 is a trivial self-fulfilling conditional, meaning that it cannot be falsified [Beierle *et al.*, 2024b], thus imposing a strong restriction on possible splitting benefits for inference. We develop a generalization of this safety condition, allowing the intersection of Δ_1 and Δ_2 to contain more meaningful conditionals. This greatly broadens the application possibilities of syntax splitting by increasing both the amount of splittings and the amount of belief bases where splittings can be exploited for inductive reasoning. The main contributions of this paper are:

- Generalization of safe conditional syntax splitting for belief bases;
- Identification of the subclass of genuine splittings, separating them from the large class of simple splittings that have no benefits for inductive inference because existing postulates cannot be meaningfully applied to them;
- Adapted postulates (CRel^{g}), (CInd^{g}), and ($\text{CSynSplit}^{\text{g}}$) for generalized safe conditional syntax splitting;
- Showing that ($\text{CSynSplit}^{\text{g}}$) implies (CSynSplit), but not the other way around;
- Evaluation of several inductive inference operators with respect to generalized safe conditional syntax splitting ($\text{CSynSplit}^{\text{g}}$).

After recalling the needed background in Sec. 2, we point out the limitations of safe conditional splittings (Sec. 3), introduce generalized and genuine splittings (Sec. 4), adapt the postulates for inference (Sec. 5), and evaluate inductive inference operators (Sec. 6) before concluding in Sec. 7.

2 Formal Basics

Let $\mathcal{L}$ be a finitely generated propositional language over a signature Σ with atoms $a, b, c, \dots$ and with formulas $A, B, C, \dots$ We may write AB instead of $A \wedge B$, and overline formulas to indicate negation, i.e., $\overline{A}$ means $\neg A$. Let Ω denote the set of *possible worlds* over $\mathcal{L}$, taken here simply as the set of all propositional interpretations over $\mathcal{L}$. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in $\omega \in \Omega$; in this case ω is called a *model* of A, and the set of all models of A is denoted by $\mathit{Mod}(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $\mathit{Mod}(A) \subseteq \mathit{Mod}(B)$, as usual. We will use ω both for the model and the corresponding complete conjunction of all positive or negated atoms, allowing us to use ω both as an interpretation and a proposition.

For $\Theta \subseteq \Sigma$, let $\mathcal{L}(\Theta)$ or short $\mathcal{L}_\Theta$ denote the propositional language defined by Θ, with associated set of interpretations $\Omega(\Theta)$ or short Ω_Θ. Note that while each formula of $\mathcal{L}(\Theta)$ can also be considered as a formula of $\mathcal{L}$, the interpretations $\omega^\Theta \in \Omega(\Theta)$ are not elements of $\Omega(\Sigma)$ if $\Theta \neq \Sigma$. But each interpretation $\omega \in \Omega$ can be written uniquely in the form $\omega = \omega^\Theta \omega^{\overline{\Theta}}$ with concatenated $\omega^\Theta \in \Omega(\Theta)$ and $\omega^{\overline{\Theta}} \in \Omega(\overline{\Theta})$, where $\overline{\Theta} = \Sigma \backslash \Theta$. The world ω^Θ is called the *reduct* of ω to Θ [Delgrande, 2017]. If $\Omega' \subseteq \Omega$ is a subset of models, then $\Omega'|_\Theta = \{\omega^\Theta | \omega \in \Omega'\} \subseteq \Omega(\Theta)$ restricts Ω' to a subset of $\Omega(\Theta)$. In the following, we will often denote subsignatures of Σ by $\Sigma_1, \Sigma_2, \dots$ and write ω^i instead of ω^{Σ_i} to ease notation.

By making use of a conditional operator $|$, we introduce the language $(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. Conditionals $(B|A)$ are meant to express plausible, defeasible rules "If A then plausibly (usually, possibly, probably, typically etc.) B". For a world ω a conditional $(B|A)$ is either *verified* by ω if $\omega \models AB$, *falsified* by ω if $\omega \models A\overline{B}$, or *not applicable* to ω if $\omega \models \overline{A}$. A conditional $(F|E)$ is called *self-fulfilling*, or *trivial*, if $E \models F$, i.e., there is no world that can falsify it. A popular semantic framework for interpreting conditionals are *ordinal conditional functions (OCFs)* $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$. OCFs, also called *ranking functions*, introduced, in a more general form, by [Spohn, 1988]. Intuitively, less plausible worlds are assigned higher numbers. Formulas are assigned the rank of their most plausible models, i.e., $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$. The rank of $(B|A)$ is $\kappa(B|A) = \kappa(AB) - \kappa(A)$. A conditional $(B|A)$ is *accepted* by κ, written as $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$, i.e., iff AB is more plausible than $A\overline{B}$. This is lifted to belief bases via $\kappa \models \Delta$ if $\kappa \models (B|A)$ for all $(B|A) \in \Delta$. Belief bases Δ (over Σ) consist of finitely many conditionals from $(\mathcal{L} \mid \mathcal{L})$. Consistency of such a belief base Δ can be defined in terms of OCFs [Pearl, 1990]: Δ is (strongly) consistent iff there is an OCF κ such that $\kappa \models \Delta$ and $\kappa(\omega) < \infty$ for all $\omega \in \Omega$. We focus on (strongly) consistent belief bases in the sense of [Pearl, 1990; Goldszmidt and Pearl, 1996] in order to elaborate our approach without having to deal with distracting technical particularities. The nonmonotonic inference relation $\mid\!\sim_\kappa$ induced by an OCF κ is [Spohn, 1988]

$$A \mid\!\sim_\kappa B \quad \text{iff} \quad A \equiv \bot \text{ or } \kappa(AB) < \kappa(A\overline{B}). \tag{1}$$

The *marginal of* κ *on* $\Theta \subseteq \Sigma$, denoted by $\kappa|_\Theta$, is defined by $\kappa|_\Theta(\omega^\Theta) = \kappa(\omega^\Theta)$ for any $\omega^\Theta \in \Omega(\Theta)$. Here ω^Θ is treated as a world in $\kappa|_\Theta(\omega^\Theta)$ but as a formula in $\kappa(\omega^\Theta)$. Note that this marginalization is a special case of the general forgetful functor $\mathit{Mod}(\sigma)$ from Σ-models to Θ-models [Beierle and Kern-Isberner, 2012] where σ is the inclusion from Θ to Σ.

An *inductive inference operator* [Kern-Isberner *et al.*, 2020] is a mapping $\mathbf{C}$ that assigns to each belief base $\Delta \subseteq (\mathcal{L} \mid \mathcal{L})$ an inference relation $\mid\!\sim_\Delta$ on $\mathcal{L}$, i.e., $\mathbf{C} : \Delta \mapsto \mid\!\sim_\Delta$, such that the following two properties hold:

Direct Inference (DI): If $(B|A) \in \Delta$ then $A \mid\!\sim_\Delta B$, and

Trivial Vacuity (TV): $A \mid\!\sim_\emptyset B$ implies $A \models B$.

Examples for inductive inference operators are:

p-entailment $\mid\!\sim^p$ [Goldszmidt and Pearl, 1996]: Considers all ranking models of Δ; coincides with System P-inference [Lehmann and Magidor, 1992; Dubois and Prade, 1994].

System Z $\mid\!\sim^z$ [Goldszmidt and Pearl, 1996]: Uses an OCF based on the tolerance partition of Δ; coincides with rational closure [Lehmann and Magidor, 1992].

c-inference $\mid\!\sim^c$ [Beierle *et al.*, 2018; Beierle *et al.*, 2021]: Considers all c-representations of Δ [Kern-Isberner, 2001; Kern-Isberner, 2004].

System W $\mid\!\sim^{\mathsf{w}}$ [Komo and Beierle, 2020; Komo and Beierle, 2022] Also considers the tolerance partition of Δ; extends both c-inference and system Z.

3 Limitations of Conditional Syntax Splitting

Syntax splittings describe that a belief base contains completely independent information about different parts of the signature. According to [Kern-Isberner *et al.*, 2020], a belief base Δ *splits* into subbases Δ_1, Δ_2 if $\{\Sigma_1, \Sigma_2\}$ is a partition of Σ such that $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_i \subset (\mathcal{L}_i | \mathcal{L}_i)$, $\mathcal{L}_i = \mathcal{L}(\Sigma_i)$ for $i \in \{1, 2\}$, $\Sigma_1 \cap \Sigma_2 = \emptyset$, and $\Sigma_1 \cup \Sigma_2 = \Sigma$, denoted as

$$\Delta = \Delta_1 \bigcup_{\Sigma_1, \Sigma_2} \Delta_2. \tag{2}$$

Syntax splittings are very useful for formalizing the idea that independent information about different topics should not affect each other in reasoning. Syntax splittings were generalized in [Heyninck *et al.*, 2023] to *conditional* syntax splittings, which allow subbases to share some atoms in a given subsignature Σ_3.

Definition 1 ([Heyninck *et al.*, 2023])**.** *A belief base* Δ splits into subbases Δ_1,Δ_2 conditional on Σ_3*, if there are* $\Sigma_1, \Sigma_2 \subseteq \Sigma$ *such that* $\Delta_i = \Delta \cap (\mathcal{L}(\Sigma_i \cup \Sigma_3) \mid \mathcal{L}(\Sigma_i \cup \Sigma_3))$ *for* $i = 1, 2$, *and* $\{\Sigma_1, \Sigma_2, \Sigma_3\}$ *is a partition of* Σ. *This is denoted as*

$$\Delta = \Delta_1 \bigcup_{\Sigma_1, \Sigma_2} \Delta_2 \mid \Sigma_3. \tag{3}$$

Unlike syntax splitting, conditional syntax splitting does not require the subbases Δ_1 and Δ_2 to be disjoint. For the remainder of this paper, we will use the notation introduced in the following straightforward proposition.

Proposition 2. *Let* $\Delta = \Delta_1 \bigcup_{\Sigma_1, \Sigma_2} \Delta_2 \mid \Sigma_3$ *and let*

$$\Delta_3 = \Delta_1 \cap \Delta_2 \tag{4}$$

$$\Delta_{1\backslash 3} = \Delta_1 \setminus \Delta_3 \tag{5}$$

$$\Delta_{2\backslash 3} = \Delta_2 \setminus \Delta_3. \tag{6}$$

Then $\Delta_{1\backslash 3}, \Delta_{2\backslash 3}, \Delta_3$ *are pairwise disjoint and*

$$\Delta = \Delta_{1\backslash 3} \cup \Delta_{2\backslash 3} \cup \Delta_3. \tag{7}$$

Note that in Proposition 2, $\Delta_3 = \Delta \cap (\mathcal{L}(\Sigma_3)|\mathcal{L}(\Sigma_3))$, implying that $\Delta_3 \subseteq (\mathcal{L}(\Sigma_3)|\mathcal{L}(\Sigma_3))$, and, for $i \in \{1,2\}$, $\Delta_{i'\backslash 3} \subseteq (\mathcal{L}(\Sigma_i \cup \Sigma_3)|\mathcal{L}(\Sigma_i \cup \Sigma_3))$.

For $\omega \in \Omega$ and $A \in \mathcal{L}(\Sigma_i)$ we have

$$\omega^1\omega^3\omega^2 \models A \quad \text{iff} \quad \omega^i\omega^3 \models A. \tag{8}$$

Given some $\Sigma_3 \subseteq \Sigma$, every belief base has at least one syntax splitting conditional on Σ_3.

Proposition 3. *Let* Δ *be a belief base over a signature* Σ*. For every* $\Sigma_3 \subseteq \Sigma$*, there exists the conditional syntax splitting* $\Delta = \Delta_1 \bigcup_{\Sigma \setminus \Sigma_3, \emptyset} \Delta_2 \mid \Sigma_3$*.*

Note that for a splitting adhering to this form we have $\Delta_1 = \Delta$ and $\Delta_2 = \Delta_3 = \Delta \cap (\mathcal{L}(\Sigma_3)|\mathcal{L}(\Sigma_3))$.

Given a complete conjunction over Σ_3, i.e., a formula uniquely describing a world in $\Omega(\Sigma_3)$, conditional syntax splittings in general do not ensure complete independence of Δ_1 and Δ_2 (for details see [Heyninck *et al.*, 2023], Ex. 6). To fix this, *safe* conditional syntax splittings were introduced.

Definition 4 ([Heyninck *et al.*, 2023])**.** *A belief base* $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ *can be* safely split into subbases Δ_1, Δ_2 conditional on a subsignature Σ_3*, writing*

$$\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3 \tag{9}$$

if the following safety property *holds for* $i, i' \in \{1,2\}$*,* $i \neq i'$*:*

$$\begin{aligned} &\text{for every } \omega^i\omega^3 \in \Omega(\Sigma_i \cup \Sigma_3), \text{ there is } \omega^{i'} \in \Omega(\Sigma^{i'}) \\ &\text{s.t. } \omega^i\omega^3\omega^{i'} \not\models \bigvee_{(F|E)\in\Delta_{i'}} E \wedge \neg F. \end{aligned} \tag{10}$$

The safety condition demands, in essence, that no complete conjunction over Σ_3 may force the falsification of a conditional in Δ when considering Σ as a whole.

Example 5 (Δ^{sun})**.** *Consider the belief base* $\Delta^{sun} = \{(\overline{s}|r), (\overline{r}|s), (b|sr), (g|b), (o|s\overline{r}), (\overline{o}|r), (u|or)\}$ *describing the following: If it is (r)ainy, then usually it is not (s)unny and vice versa. If it is rainy and sunny at the same time, then we can usually observea rain(b)ow. Maybe superstitiously, we believe that there is usually some (g)old to be found at the end of the rainbow. Unrelated to this, we usually spend some time (o)utside if it is sunny and not rainy. If it is rainy, then we usually do not spend time outside. If, despite our normal habits, we do spend time outside and it is rainy, then we usually have an (u)mbrella.* Δ^{sun} *has a safe conditional syntax splitting*

$$\Delta^{sun} = \Delta_1^{sun} \bigcup^{\mathsf{s}}_{\{g\},\{s,r,o,u\}} \Delta_2^{sun} \mid \{b\} \tag{11}$$

where $\Sigma_1 = \{g\}, \Sigma_2 = \{s,r,o,u\}, \Sigma_3 = \{b\}, \Delta_1^{sun} = \{(g|b)\}$*,* $\Delta_2^{sun} = \{(\overline{s}|r), (\overline{r}|s), (b|sr), (o|s\overline{r}), (\overline{o}|r), (u|or)\}$*, and* $\Delta_3^{sun} = \emptyset$*. This splitting is safe: We can extend any* $\omega^1 \in \Omega(\Sigma_1 \cup \Sigma_3)$ *by any* $\omega' \in \Omega(\Sigma_2)$ *with* $\omega' \models \overline{s} \wedge \overline{r} \wedge \overline{o} \wedge \overline{u}$ *without falsifying a conditional in* Δ_2^{sun}*. Similarly we can extend any* $\omega^2 \in \Omega(\Sigma_2 \cup \Sigma_3)$ *by any* $\omega'' \in \Omega(\Sigma_1)$ *with* $\omega'' \models g$ *without falsifying a conditional in* Δ_1^{sun}*.*

Safe conditional syntax splitting provides similar benefits for inductive inference as syntax splitting. Reasoning in the language of Δ_1 is independent of the conditionals in Δ_2, and vice versa, given we have full knowledge over the atoms in Σ_3. However, it has been shown recently that the safety property (10) imposes a strong, undesired restriction on Δ_3.

Lemma 6 ([Beierle *et al.*, 2024b])**.** *Let* $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$*, then* $\Delta_3 = \Delta_1 \cap \Delta_2$ *contains only self-fulfilling conditionals.*

While it is true that Δ_3 can not contain "meaningful" information, the elements in Σ_3 are still relevant and can occur in conditionals of both Δ_1 and Δ_2.

A generalization of the safety property to avoid the effect described in Lemma 6 would be advantageous. Recall that Σ_3 represents a sort of global knowledge, that should be considered in both subbases. However it is not always possible to find a safe splitting, given some intuitive or in practice desirable allocation of signature elements to Σ_3.

Example 7 (Δ^{sun} cont.)**.** *Assume we want to reason based on* Δ^{sun}*, under the assumption that we have full knowledge about* s *and* r*. A conditional syntax splitting reflecting our knowledge about the weather is*

$$\Delta^{sun} = \Delta_1^{sun} \bigcup_{\{b,g\},\{o,u\}} \Delta_2^{sun} \mid \{s,r\} \tag{12}$$

where $\Sigma_1 = \{b,g\}, \Sigma_2 = \{o,u\}, \Sigma_3 = \{s,r\}, \Delta_1^{sun} = \{(\overline{s}|r), (\overline{r}|s), (b|sr), (g|b)\}, \Delta_2^{sun} = \{(\overline{s}|r), (\overline{r}|s), (o|s\overline{r}), (\overline{o}|r), (u|or)\}$*, and* $\Delta_3^{sun} = \{(\overline{s}|r), (\overline{r}|s)\}$*. Assume that we know that it is sunny and rainy at the same time, and we would like to know if there will usually be a rainbow, i.e., whether* $sr \mid\!\sim_{\Delta^{sun}} b$ *holds. Employing the splitting* (12)*, it suffices to consider* Δ_1^{sun} *to answer this query because we have full knowledge about* $\{s,r\}$*. However, because the conditionals in* Δ_3^{sun} *are not self-fulfilling the splitting* (12) *is not safe.*

Comparing the splittings (11) and (12), we can see that there are situations where (12) provides benefits not provided by (11). For instance, as it will be shown formally in the following sections, answering the query $sr \mid\!\sim_{\Delta^{sun}} b$ can be done using the subbase Δ_1^{sun} from (12) while the splitting (11) does not provide any advantage for answering this query.

Another limitation of safe conditional syntax splittings is that there exist belief bases where all safe conditional syntax splittings involve a subset relationship between the subbases.

Example 8 (Δ^{rain})**.** *Starting from* Δ^{sun}*, we get rid of our superstitious beliefs by removing the signature element* g *and all associated conditionals, yielding* $\Delta^{rain} = \{(\overline{s}|r), (\overline{r}|s), (b|sr), (o|s\overline{r}), (\overline{o}|r), (u|or)\}$*.* Δ^{rain} *has a splitting conditional on* $\{s,r\}$

$$\begin{aligned} &\Delta^{rain} = \{(\overline{s}|r), (\overline{r}|s), (b|sr)\} \\ &\bigcup_{\{b\},\{o,u\}} \{(\overline{s}|r), (\overline{r}|s), (o|s\overline{r}), (\overline{o}|r), (u|or)\} \mid \{s,r\} \end{aligned} \tag{13}$$

which, however, is not safe. In fact, every safe splitting of $\Delta^{rain} = \Delta_1^{rain} \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2^{rain} \mid \Sigma_3$ *satisfies* $\Delta_1^{rain} \subseteq \Delta_2^{rain}$ *or* $\Delta_2^{rain} \subseteq \Delta_1^{rain}$*.*

Every conditional syntax splitting $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ with $\Delta_1 \subseteq \Delta_2$ or $\Delta_2 \subseteq \Delta_1$ is of little use for inductive inference. Suppose $\Delta_1 \subseteq \Delta_2$. Then answering any query over

$\Sigma_2 \cup \Sigma_3$ requires considering Δ as a whole because $\Delta_2 = \Delta$. Furthermore, any query over $\Sigma_1 \cup \Sigma_3$ can also not benefit from the splitting. This is because atoms of Σ_1 can not appear in Δ_1, since all conditionals of Δ_1 are defined over Σ_3 as $\Delta_1 = \Delta_1 \cap \Delta_2 = \Delta_3$ and full knowledge of Σ_3 is required to make use of the splitting.

The observations above give rise to two points. First, we will extend the notion of safety to cover conditional splittings like (12) and (13). Second, we will identify splittings that are useful for inductive inference.

4 Generalized Safe Conditional Syntax Splitting

We first introduce *generalized safe* splittings as a generalization of safe splittings to cover cases where the subbases may share non-trivial conditionals, and we introduce *genuine* splittings, which identify splittings that provide benefits for inductive inference.

Definition 9. *A belief base* $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ *can be* generalized safely split into subbases Δ_1, Δ_2 conditional on a subsignature Σ_3*, writing*

$$\Delta = \Delta_1 \textstyle\bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3 \tag{14}$$

if the following generalized safety property *holds for* $i, i' \in \{1, 2\}, i \neq i'$*:*

$$\begin{aligned} &\textit{for every } \omega^i\omega^3 \in \Omega(\Sigma_i \cup \Sigma_3), \textit{ there is } \omega^{i'} \in \Omega(\Sigma_{i'}) \\ &\textit{s.t. } \omega^i\omega^3\omega^{i'} \not\models \bigvee_{(F|E)\in\Delta_{i'\setminus 3}} E \wedge \neg F. \end{aligned} \tag{15}$$

The deciding difference in (15) compared to (10) is that only conditionals in $\Delta_{i'\setminus 3}$ are considered for the generalized safety property as opposed to all conditionals in $\Delta_{i'}$ for the safety property.

Proposition 10. *Let Δ be a belief base over Σ with a conditional syntax splitting* $S : \Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$.

1. *If S is safe, S is generalized safe.*
2. *If $\Delta_3 = \Delta_1 \cap \Delta_2 = \emptyset$, S is safe iff S is generalized safe.*

Generalized safety allows for more splittings adhering to a notion of safety. For example, the conditional syntax splittings in Proposition 3 are always generalized safe. Thus, for every $\Sigma_3 \subseteq \Sigma$, there always exists a generalized safe syntax splitting conditional on Σ_3. Observe that generalized safe splittings allow for non-trivial conditionals in Δ_3.

Example 11 ($\Delta^{sun}, \Delta^{rain}$ cont.)**.** *While not safe, the conditional syntax splittings in Examples 7 and 8 are generalized safe. For instance, in both examples, the conditional $(\overline{r}|s)$ can be falsified by $rs \in \Omega(\Sigma_i \cup \Sigma_3)$, thus making the splittings not safe, but since $(\overline{r}|s) \in \Delta_3^{sun}$ and $(\overline{r}|s) \in \Delta_3^{rain}$, this fact does not lead to a violation of generalized safety.*

With Example 11 and Proposition 10 we can see that there exists more generalized safe conditional syntax splittings than safe conditional syntax splittings. Thus, generalized safety properly extends the amount of belief bases that can be conditionally split while adhering to a notion of safety.

For governing inductive inference, we are only interested in (generalized) safe conditional syntax splittings. Example 8 shows that there exists belief bases that have safe conditional syntax splittings, but some Δ_i is a subset of Δ_j and therefore, even though the splitting is safe, it does not provide any meaningful information for inductive inference. We now identify splittings that are meaningful with respect to inductive inference as so-called *genuine splittings*.

Definition 12. *Let Δ be a belief base over a signature Σ. A conditional syntax splitting $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ of Δ is called* genuine*, if $\Delta_1 \not\subseteq \Delta_2$ and $\Delta_2 \not\subseteq \Delta_1$*

Note that genuine splittings can be equivalently characterized by $\Delta_{1\setminus 3} \neq \emptyset$ and $\Delta_{2\setminus 3} \neq \emptyset$. Intuitively, we call a splitting genuine if each subbase contains information that can not be found in the other subbase. Especially, the following types of splittings are not genuine.

⟨trivial⟩ (i) $\Delta = \Delta \bigcup^{\mathsf{s}}_{\Sigma,\emptyset} \emptyset \mid \emptyset$ or (ii) $\Delta = \Delta \bigcup^{\mathsf{gs}}_{\emptyset,\emptyset} \Delta \mid \Sigma$.

⟨set-empty⟩ $\Delta_1 = \emptyset$ or $\Delta_2 = \emptyset$.

⟨sig-empty⟩ $\Sigma_1 = \emptyset$ or $\Sigma_2 = \emptyset$.

While the ⟨trivial⟩ splitting (i) is safe, the splitting (ii) is not safe, unless Δ contains self-fulfilling conditionals only. However, splitting (ii) is generalized safe. Furthermore, in the case that Σ contains no elements that do not appear in Δ, the not genuine splittings are exactly the ⟨sig-empty⟩ splittings. If $\Delta_3 = \emptyset$, then the not genuine splittings are exactly the ⟨set-empty⟩ splittings. We give an example to illustrate the importance of identifying genuine splittings.

Example 13 (Δ^{rain} cont.)**.** *We continue Example 8. The belief base Δ^{rain} has a total of 37 conditional syntax splittings, out of which 32 are generalized safe splittings, but only 16 are safe splittings. Only 5 of the 36 splittings are genuine. For this belief base all genuine splittings are generalized safe, while no safe splitting is genuine.*

While in Example 13 the set of genuine and generalized safe splittings coincide, this does not hold in general. Example 13 shows that there exist belief bases for which no genuine, safe conditional syntax splitting exists, but a genuine, generalized safe splitting exists. Indeed, conditional syntax splittings that are both genuine and generalized safe are those splittings, where the properties of conditional relevance and conditional independence for inductive inference may be meaningfully applied.

5 Inductive Inference Respecting Generalized Safe Conditional Syntax Splitting

We will now introduce postulates to evaluate inductive inference operators with respect to generalized safe conditional syntax splitting. For this, we adjust the notions of conditional relevance and conditional independence from [Heyninck *et al.*, 2023] to include also generalized safe splittings. The idea of these postulates is that for a belief base with (generalized safe conditional) syntax splitting, inference over one subsignature should be independent from the information about the other subsignature (given that the valuation of the shared subsignature is fixed).

(CRel$^{\mathrm{g}}$) (adapted from [Heyninck *et al.*, 2023]) An inductive inference operator $\mathbf{C}$ satisfies (CRel$^{\mathrm{g}}$) if for any $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, for $i \in \{1,2\}$ and any $A, B \in \mathcal{L}(\Sigma_i)$, and a complete conjunction $E \in \mathcal{L}(\Sigma_3)$,

$$AE \mathrel{|\!\sim}_{\Delta} B \quad \text{iff} \quad AE \mathrel{|\!\sim}_{\Delta_i} B.$$

Thus, (CRel$^{\mathrm{g}}$) restricts the scope of inference by requiring that inferences in the sub-language $\Sigma_1 \cup \Sigma_3$ can be made taking only Δ_1 into account.

(CInd$^{\mathrm{g}}$) (adapted from [Heyninck *et al.*, 2023]) An inductive inference operator $\mathbf{C}$ satisfies (CInd$^{\mathrm{g}}$) if for any $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, for $i, j \in \{1,2\}$, $j \neq i$, and any $A, B \in \mathcal{L}(\Sigma_i)$, $D \in \mathcal{L}(\Sigma_j)$, and a complete conjunction $E \in \mathcal{L}(\Sigma_3)$, such that $DE \not\mathrel{|\!\sim}_{\Delta} \bot$ we have

$$AE \mathrel{|\!\sim}_{\Delta} B \quad \text{iff} \quad AED \mathrel{|\!\sim}_{\Delta} B.$$

Thus, an inductive inference operator satisfies (CInd$^{\mathrm{g}}$) if, for any Δ that safely splits into Δ_1 and Δ_2 conditional on Σ_3, whenever we have all the necessary information about Σ_3, inferences from one sub-language are independent from formulas over the other sub-language.

Syntax splitting (CSynSplit$^{\mathrm{g}}$) is the combination of the two properties (CInd$^{\mathrm{g}}$) and (CRel$^{\mathrm{g}}$).

(CSynSplit$^{\mathrm{g}}$) (adapted from [Heyninck *et al.*, 2023]) An inductive inference operator $\mathbf{C}$ satisfies *conditional syntax splitting* (CSynSplit$^{\mathrm{g}}$) if it satisfies (CRel$^{\mathrm{g}}$) and (CInd$^{\mathrm{g}}$).

The difference between (CSynSplit) and our new variant (CSynSplit$^{\mathrm{g}}$) is that (CSynSplit) was defined regarding safe conditional syntax splittings only, while our adjusted variant (CSynSplit$^{\mathrm{g}}$) takes into account all generalized safe conditional syntax splittings. Thus, an inductive inference operator satisfing (CSynSplit$^{\mathrm{g}}$) respects an increased number of conditional splittings.

Example 14 (Δ^{rain} cont.)**.** *Recall the splitting* (13) *for* Δ^{rain} *from Example 8. Let* $\mathbf{C} : \Delta \mapsto \mathrel{|\!\sim}^{rain}_{\Delta}$ *be an inductive inference operator that satisfies (CSynSplit$^{\mathrm{g}}$). Applying (CRel$^{\mathrm{g}}$), we obtain that the inference* $sr \mathrel{|\!\sim}^{rain}_{\Delta} b$ *holds iff the inference* $sr \mathrel{|\!\sim}^{rain}_{\Delta_1} b$ *holds. Thus, if we want to know whether the inference holds in* Δ*, it is sufficient to consider only* Δ_1*, reducing the amount of conditionals we need to take into account from 6 to 3. By applying (CInd$^{\mathrm{g}}$), we additionally know that the inferences* $sro \mathrel{|\!\sim}^{rain}_{\Delta} b$ *and* $sr\overline{o} \mathrel{|\!\sim}^{rain}_{\Delta} b$ *hold if the inference* $sr \mathrel{|\!\sim}^{rain}_{\Delta} b$ *holds. In this way, we can localize our reasoning tasks for the entire belief base to a smaller subbase, given that our reasoning mechanism satisfies (CSynSplit$^{\mathrm{g}}$).*

The postulate (CSynSplit$^{\mathrm{g}}$) covers and indeed properly generalizes (CSynSplit):

Proposition 15. *The following relationships hold:*

1. *(CSynSplit$^{\mathrm{g}}$) implies (CSynSplit).*
2. *(CSynSplit) does not imply (CSynSplit$^{\mathrm{g}}$).*

While the first part of Proposition 15 holds trivially, the second part can be shown by constructing an inductive inference operator satisfying (CSynSplit) but violating (CSynSplit$^{\mathrm{g}}$). Such an operator can be obtained by applying an inference operator satisfying (CSynSplit) if the input belief base Δ has a genuine safe conditional syntax splitting, and an inference operator violating (CSynSplit$^{\mathrm{g}}$) otherwise.

6 Evaluating Inductive Inference Operators with respect to (CSynSplit$^{\mathrm{g}}$)

In this section we consider several inductive inference operators and evaluate whether they satisfy (CSynSplit$^{\mathrm{g}}$).

6.1 System Z and System W

System Z is an OCF-based inductive inference operator based on the ranking function κ^z [Pearl, 1990]. The definition of κ^z crucially relies on the notion of *tolerance*. A conditional $(B|A)$ is *tolerated* by a set of conditionals $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ if there is a world $\omega \in \Omega$ such that $\omega \models AB$ and $\omega \models \bigwedge_{i=1}^{n}(\overline{A_i} \vee B_i)$, i.e., iff ω verifies $(B|A)$ and does not falsify any conditional in Δ. For every consistent knowledge base, the notion of tolerance yields a unique *inclusion-maximal ordered partition*, in the following denoted by $OP(\Delta) = (\Delta^0, \ldots, \Delta^k)$, of Δ where each Δ_i is the (with respect to set inclusion) maximal subset of $\bigcup_{j=i}^{k} \Delta_j$ that is tolerated by $\bigcup_{j=i}^{k} \Delta_j$. Intuitively, general conditionals of Δ are placed in the first sets of $OP(\Delta)$ while more specific conditionals are placed in later parts of the partition. The system Z ranking function $\kappa^z(\omega)$ is defined as follows. If ω does not falsify any conditional in Δ, then let $\kappa^z(\omega) = 0$. Otherwise, let Δ_j be the latest part in the tolerance partition containing a conditional falsified by ω, and let $\kappa^z(\omega) = j+1$ [Goldszmidt and Pearl, 1996]. System Z yields the inference relation induced by κ^z, i.e., $\mathbf{C^z} : \Delta \mapsto \mathrel{|\!\sim}_{\kappa^z}$.

Just like safe splittings, generalized safe splittings are respected by the notion of tolerance.

Proposition 16. *Let* $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$*. Then, for any* $i \in \{1,2\}$*,* Δ_i *tolerates* $(B|A) \in \Delta_i$ *iff* Δ *tolerates* $(B|A)$*.*

Proof. First, assume $(B|A)$ is tolerated by Δ_i. Then there must be some $\omega^i\omega^3$ such that $\omega^i\omega^3 \models AB$ and there is no $(D|C) \in \Delta_i$ such that $\omega^i\omega^3 \models C\overline{D}$. In particular, there is no such conditional in Δ_3. Due to generalized safe ty there is an extension $\omega^{i'}$ such that there is no conditional $(F|E) \in \Delta_{i'\backslash 3}$ with $\omega^i\omega^3\omega^{i'} \models E\overline{F}$. Since $\Delta = \Delta_i \cup \Delta_{i'\backslash 3}$, $(B|A)$ is tolerated by Δ. The other direction is immediate. □

For System Z we can then show the following result.

Proposition 17. *System Z satisfies (CRel$^{\mathrm{g}}$), but does not satisfy (CInd$^{\mathrm{g}}$) and thus does not satisfy (CSynSplit$^{\mathrm{g}}$).*

Crucial to showing that System Z satisfies (CRel$^{\mathrm{g}}$) is Proposition 16 which can be used to show that the ordered partition of Δ_i is respected by Δ and vice versa. The fact that System Z does not satisfy (CInd$^{\mathrm{g}}$) follows from the fact, that it does not satisfy (CInd) [Heyninck *et al.*, 2023].

System W is an inference operator also using the tolerance partition $OP(\Delta)$, but while System Z considers only which parts of $OP(\Delta)$ contain falsified conditionals, system W also takes into account the structural information about which conditionals are falsified [Komo and Beierle, 2020; Komo and Beierle, 2022]. For this, System W uses the *preferred structure on worlds* $<^w_{\Delta}$ which compares worlds according to the set of conditionals in Δ they falsify, giving preference to the more specific conditionals according to $OP(\Delta)$. Using this ordering, B is a system W inference from A, denoted $A \mid\!\sim^{w}_{\Delta} B$, if for every $\omega' \in Mod_{\Sigma}(A\overline{B})$ there is an $\omega \in Mod_{\Sigma}(AB)$ with $\omega <^{w}_{\Delta} \omega'$. For details we refer to [Komo and Beierle, 2020; Komo and Beierle, 2022].

Proposition 18. *System W satisfies (CRel^{g}) and (CInd^{g}) and thus (CSynSplit^{g}).*

The proof of Proposition 18 is based on the observations that for $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ the order $\omega <^{\mathsf{w}}_{\Delta} \omega'$ of two worlds coinciding on $\Sigma_2 \cup \Sigma_3$ depends only on Δ_1 and that the order $\omega <^{\mathsf{w}}_{\Delta_1} \omega'$ of worlds induced by Δ_1 does not change if we change the valuation over Σ_2 in the worlds.

6.2 Inference with Single c-Representations

Among the OCF models of Δ, c-representations are special ranking models obtained by assigning individual integer impacts to the conditionals in Δ and generating the world ranks as the sum of impacts of falsified conditionals.

Definition 19 (c-representation [Kern-Isberner, 2001; Kern-Isberner, 2004])**.** *A* c-representation *of* $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ *is an OCF* κ *constructed from non-negative impacts* $\eta_j \in \mathbb{N}_0$ *assigned to each* $(B_j|A_j)$ *such that* κ *accepts* Δ *and is given by:*

$$\kappa(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j\overline{B}_j}} \eta_j \tag{16}$$

c-Representations can conveniently be specified using a constraint satisfaction problem (for detailed explanations, see [Kern-Isberner, 2001; Kern-Isberner, 2004]):

Definition 20 ($CR(\Delta)$, [Kern-Isberner, 2001; Beierle *et al.*, 2018])**.** *The* constraint satisfaction problem $CR(\Delta)$ for c-representations of $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ *is given by the conjunction of the constraints, for all* $j \in \{1, \dots, n\}$*:*

$$\eta_j \geqslant 0 \tag{17}$$

$$\eta_j > \min_{\omega \models A_j B_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k - \min_{\omega \models A_j \overline{B}_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k \tag{18}$$

Note that (17) expresses that falsification of conditionals should make worlds not more plausible, and (18) ensures that κ as specified by (16) accepts Δ. A solution of $CR(\Delta)$ is a vector $\vec{\eta} = (\eta_1, \dots, \eta_n)$ of natural numbers. $Sol(CR(\Delta))$ denotes the set of all solutions of $CR(\Delta)$. For $\vec{\eta} \in Sol(CR(\Delta))$ and κ as in Equation (16), κ is the *OCF induced by* $\vec{\eta}$ and is denoted by $\kappa_{\vec{\eta}}$. $CR(\Delta)$ is sound and complete [Kern-Isberner, 2001; Beierle *et al.*, 2018]: For every $\vec{\eta} \in Sol(CR(\Delta))$, $\kappa_{\vec{\eta}}$ is a c-representation with $\kappa_{\vec{\eta}} \models \Delta$, and for every c-representation κ with $\kappa \models \Delta$, there is $\vec{\eta} \in Sol(CR(\Delta))$ such that $\kappa = \kappa_{\vec{\eta}}$. For $\vec{\eta}$, we will simply write $\vec{\eta}^1$ and $\vec{\eta}^2$ for the corresponding projections $\vec{\eta}|_{\Delta_1}$ and $\vec{\eta}|_{\Delta_2}$, and $(\vec{\eta}^1, \vec{\eta}^2)$ for their composition.

A fundamental property of c-representations is that for any syntax splitting $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ the composition of *any* impact vectors for the subbases yields an impact vector for Δ, and vice versa [Kern-Isberner *et al.*, 2020]. This property was also shown to extend to safe conditional syntax splittings [Beierle *et al.*, 2024b]. However, a key part in showing this was Lemma 6 which no longer holds for generalized safe conditional syntax splittings. Indeed the composition property no longer holds, as the impacts assigned to the conditionals in Δ_3 can be vastly different between the two subbases. Thus, we show a slightly weaker property here. While it still states that any impact vector for Δ can be split into impact vectors for the subbases, impact vectors for the subbases may only yield an impact vector for Δ if they match on the impacts assigned to the conditionals in Δ_3.

Proposition 21. *Let* $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$*. The following two properties hold for* $i, i' \in \{1, 2\}, i \neq i'$*:*

- *For every* $\vec{\eta} \in Sol(CR(\Delta))$ *there are* $\vec{\eta}^i \in Sol(CR(\Delta_i))$, $\vec{\eta}^{i'} \in Sol(CR(\Delta_{i'}))$ *and* $\vec{\eta}^3 \in Sol(CR(\Delta_3))$ *with* $\vec{\eta}^i|_{\Delta_3} = \vec{\eta}^{i'}|_{\Delta_3} = \vec{\eta}^3$*, such that* $\vec{\eta} = (\vec{\eta}^i|_{\Delta_{i\setminus 3}}, \vec{\eta}^{i'}|_{\Delta_{i'\setminus 3}}, \vec{\eta}^3)$.
- *For every* $\vec{\eta}^i \in Sol(CR(\Delta_i)), \vec{\eta}^{i'} \in Sol(CR(\Delta_{i'}))$ *and* $\vec{\eta}^3 \in Sol(CR(\Delta_3))$ *with* $\vec{\eta}^i|_{\Delta_3} = \vec{\eta}^{i'}|_{\Delta_3} = \vec{\eta}^3$ *there is* $\vec{\eta} \in Sol(CR(\Delta))$ *such that* $\vec{\eta} = (\vec{\eta}^i|_{\Delta_{i\setminus 3}}, \vec{\eta}^{i'}|_{\Delta_{i'\setminus 3}}, \vec{\eta}^3)$.

Due to the condition that $\vec{\eta}^i|_{\Delta_3} = \vec{\eta}^{i'}|_{\Delta_3} = \vec{\eta}^3$, we have that $\vec{\eta}^i = (\vec{\eta}^i|_{\Delta_3}, \vec{\eta}^3)$ and $\vec{\eta}^{i'} = (\vec{\eta}^{i'}|_{\Delta_3}, \vec{\eta}^3)$. And thus $\vec{\eta} = (\vec{\eta}^i, \vec{\eta}^{i'}|_{\Delta_{i'\setminus 3}})$ for both points of Proposition 21.

Example 22. *For the belief base* $\Delta = \{(c|a), (b|a)\}$*, a possible c-representation* $\kappa_{\vec{\eta}}$ *is given by* $\vec{\eta}_1 = (1, 1) \in Sol(CR(\Delta))$*, yielding the following OCF:*

$$\kappa(abc) = \kappa(\overline{a}bc) = \kappa(\overline{a}\overline{b}c) = \kappa(\overline{a}b\overline{c}) = \kappa(\overline{a}\overline{b}\overline{c}) = 0,$$
$$\kappa(ab\overline{c}) = \kappa(a\overline{b}c) = 1, \kappa(a\overline{b}\overline{c}) = 2.$$

Because $\Delta = \{(b|a)\} \bigcup^{\mathsf{gs}}_{\{b\},\{c\}} \{(c|a)\} \mid \{a\}$ *and* $\Delta_3 = \emptyset$ *we can obtain* $\vec{\eta}_1$ *by combining* $\vec{\eta}_1^1 = (1)$ *and* $\vec{\eta}_1^2 = (1)$ *and vice versa utilizing Proposition 21.*

To show that nonmonotonic reasoning with c-representations satisfies (CSynSplit$^{\mathsf{g}}$) we employ the concept of conditional κ-independence.

Definition 23 ([Heyninck *et al.*, 2023],[Spohn, 2012])**.** *Let* $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq \Sigma$ *where* Σ_1, Σ_2 *and* Σ_3 *are pairwise disjoint and let* κ *be an OCF.* Σ_1, Σ_2 *are* conditionally κ-independent given Σ_3*, in symbols* $\Sigma_1 \perp\!\!\!\perp_{\kappa} \Sigma_2 | \Sigma_3$*, if for all* $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$*, and* $\omega^3 \in \Omega(\Sigma_3)$*, it holds that* $\kappa(\omega^1|\omega^2\omega^3) = \kappa(\omega^1|\omega^3)$.

Proposition 24 (adapted from [Beierle *et al.*, 2024b])**.** *Let* $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$*, and* κ *a c-representation with* $\kappa \models \Delta$*. Then* $\Sigma_1 \perp\!\!\!\perp_{\kappa} \Sigma_2 | \Sigma_3$.

Now we introduce an alternative characterization of (CInd) and (CRel) based on OCFs, adapted to our new notion of generalized safe splittings.

Proposition 25 (adapted from [Heyninck *et al.*, 2023])**.** *An inductive inference operator for OCFs* $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_{\Delta}$ *satisfies (CInd^{g}) iff for any* $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ *we have* $\Sigma_1 \perp\!\!\!\perp_{\kappa_{\Delta}} \Sigma_2 | \Sigma_3$.

Proposition 26 (adapted from [Heyninck *et al.*, 2023]). *An inductive inference operator for OCFs* $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ *satisfies (CRel*^{g}*) iff for any* $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ *and* $i \in \{1,2\}$ *we have* $\kappa_{\Delta_i} = \kappa_\Delta|_{\Sigma_i \cup \Sigma_3}$.

We will now define model-based inductive inference operators assigning a c-representation κ to each Δ, by employing the concept of selection strategies.

Definition 27 (selection strategy σ, [Beierle and Kern-Isberner, 2021]). *A* selection strategy (for c-representations) *is a function* $\sigma : \Delta \mapsto \vec{\eta}$ *assigning to each conditional belief base* Δ *an impact vector* $\vec{\eta} \in Sol(CR(\Delta))$.

Each selection strategy yields an inductive inference operator $\mathbf{C}^{c\text{-}rep}_\sigma : \Delta \mapsto \kappa_{\sigma(\Delta)}$ where $\mathrel{|\!\sim}_{\kappa_{\sigma(\Delta)}}$ is obtained via Equation (1) from $\kappa_{\sigma(\Delta)}$. Note that each $\mathrel{|\!\sim}_{\kappa_{\sigma(\Delta)}}$ satisfies both (Direct Inference) and (Trivial Vacuity). A recent example for a specific selection strategy are *minimal core c-representations* [Wilhelm *et al.*, 2024].

In principle, for every Δ, a selection strategy may choose some impact vector independently from the choices for all other belief bases. The following property characterizes selection strategies that preserve the impacts chosen for subbases if Δ splits into these subbases.

(IP-CSP^g) (adapted from [Beierle *et al.*, 2024b]) A selection strategy σ is *impact preserving w.r.t. conditional belief base splitting* if, for every generalized safe conditional belief base splitting $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, for $i \in \{1,2\}$, we have $\sigma(\Delta_i) = \sigma(\Delta)|_{\Delta_i}$.

In [Beierle *et al.*, 2024b] it is shown that any inductive inference operator based on an impact preserving selection strategy satisfies (CSynSplit); we extend this result to (CSynSplit^g).

Proposition 28. *Let* σ *be a selection strategy satisfying (IP-CSP*^{g}*). Then* $\mathbf{C}^{c\text{-}rep}_\sigma$ *satisfies (CRel*^{g}*) and (CInd*^{g}*) and thus (CSynSplit*^{g}*).*

After showing that inference based on a single c-representation satisfies (CSynSplit^g) if the underlying selection strategy satisfies (IP-CSP^g), we next consider inference with respect to all c-representations of a belief base.

6.3 c-Inference

c-Inference was introduced in [Beierle *et al.*, 2016; Beierle *et al.*, 2018] as the skeptical inference relation obtained by taking all c-representations of a belief base Δ into account.

Definition 29 (c-inference, $\mathrel{|\!\sim}^{c\text{-}sk}_\Delta$, [Beierle *et al.*, 2016]). *Let* Δ *be a belief base and let A, B be formulas. B is* a (skeptical) c-inference from A in the context of Δ, *denoted by* $A \mathrel{|\!\sim}^{c\text{-}sk}_\Delta B$, *iff* $A \mathrel{|\!\sim}_\kappa B$ *holds for all c-representations* κ *of* Δ*, yielding the inductive inference operator*

$$\mathbf{C}^{c\text{-}sk} : \Delta \mapsto \mathrel{|\!\sim}^{c\text{-}sk}_\Delta$$

Before proving that c-inference satisfies conditional syntax splitting, we show a lemma stating the following observations. Consider a generalized safe conditional syntax splitting of Δ into Δ_1 and Δ_2, and a c-representation $\kappa_{\vec{\eta}}$ determined by a solution vector $\vec{\eta} \in Sol(CR(\Delta))$ together with its projections $\kappa_{\vec{\eta}^1}$ and $\kappa_{\vec{\eta}^2}$ to Δ_1 and Δ_2, respectively. Then the rank of any formula F_i over the language $\mathcal{L}(\Sigma_i \cup \Sigma_3)$ of Δ_i under the projection $\kappa_{\vec{\eta}^i}$ coincides with the rank of the formula rank determined by $\kappa_{\vec{\eta}}$.

Proposition 30 (adapted from [Beierle *et al.*, 2024b]). *For any* $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$*, for all* $\vec{\eta} \in Sol(CR(\Delta))$*, and for* $i \in \{1,2\}$*,* $F_i \in \mathcal{L}(\Sigma_i \cup \Sigma_3)$*, we have* $\kappa_{\vec{\eta}}(F_i) = \kappa_{\vec{\eta}^i}(F_i)$.

The related proposition in [Beierle *et al.*, 2024b] additionally states that the rank of the formulas F_i is zero in the OCF $\kappa_{\vec{\eta}^j}$. This no longer holds for generalized safe splittings because conditionals in Δ_3 can be falsified by F_i.

Next we can show that for every generalized safe conditional syntax splitting and every solution vector for Δ_i, we can actually find matching solution vectors for $\Delta_{i'}$ and Δ_3.

Proposition 31. *Let* Δ *be a belief base with* $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$*. Then for* $i \in \{1,2\}$*, and for every* $\vec{\eta}^i \in Sol(CR(\Delta_i))$ *there are* $\vec{\eta}^{i'} \in Sol(CR(\Delta_{i'}))$ *and* $\vec{\eta}^3 \in Sol(CR(\Delta_3))$*, such that* $\vec{\eta}^i|_{\Delta_3} = \vec{\eta}^{i'}|_{\Delta_3} = \vec{\eta}^3$.

With Propositions 21, 30, and 31 we can show:

Proposition 32. *c-Inference satisfies (CRel*^{g}*) and (CInd*^{g}*) and thus (CSynSplit*^{g}*).*

Thus also the inference taking all c-representations into account fully complies with (CSynSplit^g).

7 Conclusions and Future Work

In this paper, we generalized the notion of safety for conditional syntax splittings, allowing the subbases to share nontrivial conditionals, and thus broadened the application scope of the beneficial splitting techniques. Moreover, we identified genuine splittings as the subclass of meaningfull conditional syntax splittings. We showed that this notion of safety properly generalizes the previous notion of safety and and gave illustrative examples of belief bases that have no safe conditional syntax splitting, but a generalized safe conditional syntax splitting. We adjusted postulates for conditional syntax splitting based on the generalized notion and showed that System W and inductive inference with a single c-representation based on a selection strategy, as well as inference with all c-representations fully comply with this new property. While System Z fails to satisfy syntax splittings, we showed that it fulfills (CRel^g). Furthermore, we showed that, while (CSynSplit^g) implies (CSynSplit), the other direction does not hold.

In future work, we will investigate further inference operators like lexicographic inference [Lehmann, 1995] with respect to (CSynSplit^g), and we will study the exact relationship of our approach to syntactic contextual filtering [Dupin de Saint-Cyr and Bisquert, 2024] and to propositional forgetting [Lang *et al.*, 2003; Sauerwald *et al.*, 2022; Sauerwald *et al.*, 2024]. We will exploit the beneficial properties of splitting techniques for implementations of inductive inference [Beierle *et al.*, 2024a] and we will adapt the concepts shown here to include also belief bases that satisfy a weaker notion of consistency (cf. [Haldimann *et al.*, 2023; Haldimann *et al.*, 2024]).

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