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Revisiting Proportional Allocation with Subsidy: Simplification and Improvements

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Abstract

In this paper, we revisit the problem of fair allocation with subsidy. We first consider the allocation of m indivisible chores to n agents with additive (dis)utility functions. Under the assumption that the maximum (dis)utility of an item can be compensated by one dollar, Wu et al. (WINE 2023) showed that a total of $n/4$ dollars suffices to guarantee a proportional allocation by rounding fractional allocations. Their subsidy guarantee is optimal when n is even. For odd n, there is still a small gap between the upper and lower bounds for the total subsidy. In this paper, we propose a much simpler algorithm for the problem, which does not require rounding fractional allocations, and achieves an optimal subsidy guarantee for all values of n. We further show that our algorithm and analysis framework can be extended to the mixture of (subjective) goods and chores, achieving the optimal subsidy guarantee.

1 Introduction

The fair division problem, with a research history dating back to 1948 [Steihaus, 1948], remains a central issue in economics, computer science, and mathematics. The problem involves fairly allocating a set of m items M among a group of n heterogeneous agents N, where an allocation is a partition of the items M into n disjoint bundles. The first and most natural fairness criterion, known as *proportionality*, was introduced and studied by Steihaus [1948]. Proportionality requires that each agent receives a bundle whose utility is at least $1/n$ of the total utility of all items. When all items are divisible, the existence of proportional allocations has been extensively studied under the frameworks of "cake cutting" and "bad cake cutting", which correspond to the cases when items have *positive utility* and *negative utility*, respectively. In recent decades, the focus of fair division research has shifted towards discrete scenarios, where items are indivisible and proportional allocations may not exist. For example, consider the instance of allocating a single item between two agents with non-zero values on the item. In the discrete setting, we distinguish between the fair allocation of goods (when all items have positive utility) and the fair allocation of chores (when all items have negative utility). When the set of items contains both goods and chores, we call the allocation instance a mixture of goods and chores.

In this paper, we explore the allocation of both indivisible goods and chores, as well as their mixtures. We use $u_i : 2^M \to \mathbb{R}$ to denote the *utility function* of agent i, and we say that agent i derives *utility* $u_i(S)$ from the bundle of items $S \subseteq M$. The proportionality of agent i is thus defined by $\frac{1}{n} \cdot u_i(M)$. In this paper, we assume that the utility functions are additive. We say that an item $e \in M$ is a good (resp., chore) for agent $i \in N$ if $u_i(e) \geq 0$ (resp., $u_i(e) \leq 0$). In the most general case, an item can be good to some agent while being a chore to another.

Relaxations of Proportionality. Since proportional allocations are not guaranteed to exist and do not admit any bounded (multiplicative) approximation ratio, several relaxations of proportionality have been proposed. One such relaxation is *proportionality up to one item* (PROP1), introduced by Conitzer et al. [2017], which requires that each agent can achieve her proportional share by either adding some good she does not own or removing some chore she owns. It has been shown that PROP1 allocations always exist and can be computed efficiently for goods [Conitzer *et al.*, 2017; Barman and Krishnamurthy, 2019], chores [Brânzei and Sandomirskiy, 2024], and the mixture of goods and chores [Aziz *et al.*, 2020]. A stronger variant, *proportionality up to any item* (PROPX), requires that the proportional share can be achieved by adding or removing any single item (for goods and chores, respectively). For the allocation of goods, Aziz et al. [2020] showed that PROPX allocations may not exist. In contrast, PROPX allocations for chores always exist and can be computed efficiently [Moulin, 2018; Aziz *et al.*, 2024b].

Fair Allocation with Subsidy. The concept of resource allocation with monetary compensation has been extensively studied in economics. Maskin [1987] was one of the first to introduce the idea of incorporating monetary subsidies into the fair division problem. In this setting, each agent may receive a subsidy $s_i \geq 0$ to eliminate the perceived unfairness, with the goal of minimizing the total subsidy required. For the allocation of goods, several studies [Halpern and Shah, 2019; Brustle *et al.*, 2020] considered the fairness notion of *envy-freeness* (which implies proportionality), demonstrating that a total subsidy of $n - 1$ dollars is sufficient to achieve envy-freeness. The subsidy setting can also be applied to the allocation of chores. Wu et al. [2023] were the first to consider this problem, showing that the same amount of subsidy $(n-1)$ is sufficient to ensure envy-freeness for chores and proposing an algorithm that computes proportional allocations with a total subsidy at most $n/4$. They also proved that the upper bound of $n/4$ is optimal when n is even. However, when n is odd, their result leaves a small gap between the lower bound of $(n^2-1)/(4n)$ and the upper bound of $n/4$. Recently, Wu and Zhou [2024] partially closed this gap for the case of $n=3$, showing that a total subsidy of $2/3$ is sufficient to guarantee proportional allocations.

1.1 Our Results

In this paper, we revisit the problem of ensuring proportionality by introducing subsidies to the agents. Our main result is summarized as follows. Throughout the whole paper, we use $\mathbf{X} = (X_1, \ldots, X_n)$ to denote an allocation of the items, $\mathbf{s} = (s_1, \ldots, s_n) \in [0,1]^n$ to denote the subsidies to the agents, and $\tau(\mathbf{s}) = \sum_{i \in N} s_i$ to denote the total subsidy.

Main Result. For the allocation of a mixture of indivisible goods and chores to a group of n agents with additive utility functions, we can compute in polynomial time a PROP1 allocation $\mathbf{X}$ and subsidies $\mathbf{s}$ such that $(\mathbf{X}, \mathbf{s})$ is proportional (meaning that $u_i(X_i) + s_i \geq \frac{1}{n} \cdot u_i(M), \forall i \in N$), where the total subsidy $\tau(\mathbf{s}) \leq n/4$ when n is even and $\tau(\mathbf{s}) \leq (n^2-1)/(4n)$ when n is odd.

As a warm-up, we first consider the allocation of chores (the primary concerned setting of existing works [Wu *et al.*, 2023; Wu and Zhou, 2024]), where all items have negative utility to all agents. As in many of the existing works that study share-based fairness notions, we show that it is without loss of generality (w.l.o.g.) to only consider IDO instances, in which all agents agree on the same ordering of items, i.e., $u_i(e_1) \geq u_i(e_2) \geq \cdots \geq u_i(e_m)$ for all $i \in N$. More importantly, we show that such a reduction preserves the upper bounds on the total subsidy and the PROP1 property of the allocation, and can be extended to the mixture of goods and chores. Given that agents share the same ordinal preference on the items in an IDO instance, we can partition the items into n bundles such that any two bundles admit a small gap in utility for all agents: in fact, a partition based on round-robin would work. Now that we have n "even" bundles, it suffices to assign each bundle to the agent requiring the minimum subsidy on it in a sequential manner, and bound the total subsidy taking the perspective of the agent who receives the last bundle. Since our algorithm does not require computing any fractional allocation and rounding, the overall algorithm is very simple (consists of 5 lines), and the analysis is much shorter than that of [Wu *et al.*, 2023].

Then we show that a similar algorithm and analysis can be applied to the allocation of a mixture of *objective* goods and chores, where each item is either a good to all agents, or a chore to all agents. We show that we can still compute a partition of items into n bundles that are even to all agents, by adapting the double round-robin algorithm [Aziz *et al.*, 2022a], which first allocates the goods using round-robin in one order of agents, and then allocates the chores under a reversed order. Then, by applying the same assignment principle and analysis for upper bounding the total subsidy, we show that the optimal subsidy guarantee can also be achieved for this setting.

Finally, we generalize the algorithms and results to the mixture of *subjective* goods and chores, where an item can be a good to some agent while being a chore to another. Instead of devising a new algorithm for this setting, we show that we can reduce any subjective mixed instance to an objective mixed instance while preserving the property of the total subsidy and of PROP1. Then by using the algorithm for the objective mixed instances as a black box, we derive our results for the subjective mixed instances.

1.2 Other Related Works

Besides the fairness criteria we have introduced, several other share-based fairness notions have been proposed for specific settings. A prominent example is *maximin share* (MMS) [Budish, 2011]. To address scenarios where agents might have different weights, Farhadi et al. [2019] introduced weighted maximin share (WMMS), while Babioff et al. [2021] proposed anyprice share (APS). Further research by Babioff and Feige explored general fair share allocation in both unweighted [Babaioff and Feige, 2022] and weighted [Babaioff and Feige, 2025] settings. More recently, Babichenko et al. [2024] introduced and studied quantile share, which measures expected utility in random allocations. Another well-studied class of fairness notions is envy-based, tracing back to [Foley, 1967]. While envy-freeness implies proportionality when valuations are additive, it's not guaranteed for indivisible items. Relaxations of envy-freeness, such as *envy-freeness up to one item* (EF1) [Lipton *et al.*, 2004] and *envy-freeness up to any item* (EFX) [Caragiannis *et al.*, 2019], have received significant attention. For a comprehensive overview of the fair allocation literature, we refer to the surveys by Amanatidis et al. [2023] and Aziz et al. [2022b].

Allocation of Chores. While chore allocations are as common as goods allocations in real-world scenarios, the problem receives less attention and usually tends to admit worse results. As we have introduced, both PROP1 and PROPX allocations are guaranteed to exist for chore allocation. While MMS allocations are not guaranteed to exist [Aziz *et al.*, 2017; Feige *et al.*, 2021], much attention focuses on approximate MMS allocations [Aziz *et al.*, 2022c; Barman and Krishnamurthy, 2020; Huang and Lu, 2021], which led to the state-of-the-art ratio of $13/11$ [Huang and Segal-Halevi, 2023]. While EFX has been considered the most popular envy-based fairness notion, its existence has only been shown for some restricted settings [Aziz *et al.*, 2023; Gafni *et al.*, 2023; Aziz *et al.*, 2024b; Zhou and Wu, 2024; Kobayashi *et al.*, 2025; Tao *et al.*, 2025; Lin *et al.*, 2025].

Mixture of Goods and Chores. A setting that generalizes both the allocations of goods and chores is the mixture of goods and chores. It is worth mentioning that many existing algorithmic techniques developed for the non-mixture settings fail to work in the mixture setting [Aziz *et al.*, 2022a; Chaudhury *et al.*, 2023; Bhaskar *et al.*, 2021], and hence the mixture setting usually demands independent analysis and scrutiny [Hosseini and Sethia, 2025]. As we have introduced before, Aziz et al. [2020] proposed an algorithm that computes PROP1 allocations in polynomial time. When strengthening PROP1 to EF1, the existence and computation of EF1 allocations have been solved by Bhaskar et al. [2021] and Aziz et al. [2022a]. There are also works that study other fairness notions for the mixed setting, e.g., for EFX [Aleksandrov and Walsh, 2020; Hosseini *et al.*, 2023a; Hosseini *et al.*, 2023b] and MMS [Feige, 2022; Cousins *et al.*, 2023; Kulkarni *et al.*, 2021].

Subsidy Setting and Beyond. While our setting assumes symmetric agents, recent studies have extended the model to the weighted setting. Wu and Zhou [2024] were among the first to investigate weighted fair allocations with subsidies, demonstrating that a total subsidy of $n/3$ is sufficient to guarantee weighted proportionality. Recently, Aziz et al. [2024a] and Elmalem et al. [2024] explored weighted envy-free allocations with subsidy for the allocation of goods. Another generalization involves extending the utility function beyond additive, for example, to monotone functions [Brustle *et al.*, 2020; Kawase *et al.*, 2024], matroid rank functions [Barman *et al.*, 2022; Goko *et al.*, 2024]. The subsidy setting has also been introduced to the house allocation problem [Choo *et al.*, 2024; Dai *et al.*, 2024]. In settings where money is treated as a divisible good, our problem shares similarities with the fair allocation of mixed divisible and indivisible items, a topic that has received considerable attention [Bei *et al.*, 2021a; Bhaskar *et al.*, 2021; Bei *et al.*, 2021b; Li *et al.*, 2023; Bu *et al.*, 2024]. For a comprehensive review of existing works on mixed fair allocation, please refer to the survey by Liu et al. [2024].

2 Preliminaries

We consider the allocation of a set of indivisible items $M = \{e_1, \dots, e_m\}$ to n agents $N = \{1, \dots, n\}$, where the items can be either goods or chores. Each agent i has an additive utility function $u_i : 2^M \to \mathbb{R}$. That is, for any bundle $S \subseteq M$, $u_i(S) = \sum_{e \in S} u_i(\{e\})$, where $u_i(\{e\})$ is the utility of item e allocated to agent i. For convenience, we use $u_i(e)$ to denote $u_i(\{e\})$. We say that an item $e \in M$ is a good (resp., chore) for some agent $i \in N$ if $u_i(e) \geq 0$ (resp., $u_i(e) \leq 0$). We assume w.l.o.g. that $u_i(e) \in [-1, 1]$ for all agent $i \in N$ and item $e \in M$. An instance is denoted by $\mathcal{I} = (M, N, \mathbf{u})$, where $\mathbf{u} = (u_1, \dots, u_n)$ represents the utility functions of all agents. An allocation for instance $\mathcal{I}$ is an n-partition $\mathbf{X} = (X_1, \dots, X_n)$ of the items M, where $X_i \cap X_j = \emptyset$ for all $i \neq j$ and $\cup_{i \in N} X_i = M$. In allocation $\mathbf{X}$, agent $i \in N$ receives bundle X_i. For convenience of notation, given any set $X \subseteq M$ and $e \in M$, we use $X + e$ and $X - e$ to denote $X \cup \{e\}$ and $X \setminus \{e\}$, respectively. For any integer $k \geq 1$, we use $[k]$ to denote the set $\{1, 2, \dots, k\}$.

Definition 2.1 (Proportionality)**.** Given an instance $\mathcal{I} = (M, N, \mathbf{u})$, the proportional share of each agent $i \in N$ is defined as $\mathsf{PROP}_i = \frac{1}{n} \cdot u_i(M)$. An allocation $\mathbf{X}$ is called proportional (PROP) if each agent receives a bundle with utility at least her proportional share, i.e., $u_i(X_i) \geq \mathsf{PROP}_i$.

Next, we define two fairness notions *envy-freeness up to one item* (EF1) and *proportionality up to one item* (PROP1). We remark that any EF1 allocation is also PROP1, for the mixture of indivisible goods and chores [Aziz *et al.*, 2022a].

Definition 2.2 (PROP1)**.** An allocation $\mathbf{X}$ is called proportional up to one item (PROP1) if for any agent $i \in N$, either

- $u_i(X_i) \geq \mathsf{PROP}_i$; or
- $u_i(X_i - e) \geq \mathsf{PROP}_i$ for some $e \in X_i$; or
- $u_i(X_i + e) \geq \mathsf{PROP}_i$ for some $e \in M \setminus X_i$.

Definition 2.3 (EF1)**.** An allocation $\mathbf{X}$ is called *envy-free up to one item* (EF1) if for any two agents $i, j \in N$, either

- $u_i(X_i) \geq u_i(X_j)$; or
- $\exists\, e \in X_i \cup X_j$ such that $u_i(X_i - e) \geq u_i(X_j - e)$.

We consider the subsidy setting, where each agent $i \in N$ is subsidized with a non-negative subsidy $s_i \geq 0$. We use vector $\mathbf{s} = (s_1, \dots, s_n)$ to denote the set of subsidies and $\tau(\mathbf{s}) = \sum_{i \in N} s_i$ to denote the total subsidy.

Definition 2.4. An outcome $(\mathbf{X}, \mathbf{s})$ consisting of an allocation $\mathbf{X}$ and subsidies $\mathbf{s} = (s_1, \dots, s_n)$ is proportional (PROP) if for any $i \in N$ we have $u_i(X_i) + s_i \geq \mathsf{PROP}_i$.

Given any instance $\mathcal{I} = (M, N, \mathbf{u})$, we aim to find a PROP outcome with a small amount of total subsidy. We remark that given any allocation $\mathbf{X}$ of some instance $\mathcal{I}$, we can turn $\mathbf{X}$ into a proportional outcome by introducing subsidies to the agents as follows:

$$s_i = \max\{\mathsf{PROP}_i - u_i(X_i), 0\}, \qquad \forall i \in N.$$

Moreover, with the allocation fixed, this is the optimal way to set the subsidies. Therefore, to describe a PROP outcome, it suffices to define the allocation $\mathbf{X}$. The subsidy to each agent is automatically determined by the above equation.

It has been proved that to bound the required subsidy for proportional allocations for chores, it suffices to consider IDO instances by implementing a reduction. We show that such a reduction can be extended to the mixed setting. We remark that Feige [2022] claimed a similar reduction for the mixed setting with respect to the fairness notion of MMS.

Specifically, given any $\mathcal{I}$, we can construct (in polynomial time) another instance that is IDO and requires at least the same amount of subsidy to guarantee proportionality as $\mathcal{I}$. We define $\mathsf{IDO}(\cdot)$ as the subroutine that takes an instance as input and outputs the corresponding IDO instance.

Definition 2.5 (IDO Instances)**.** An instance is called identical ordering (IDO) if for all agent $i \in N$ we have $u_i(e_1) \geq \cdots \geq u_i(e_m)$. For any instance $\mathcal{I} = (M, N, \mathbf{u})$, $\mathsf{IDO}(\mathcal{I})$ returns the instance $(M, N, \mathbf{u}')$ such that for any $i \in N$ and $k \in \{1, \dots, m\}$, we have $u'_i(e_k) = u_i(\sigma_i(k))$ where $\sigma_i(k) \in M$ is the k-th most valuable item under u_i.

Lemma 2.6 (IDO Reduction)**.** *Given a PROP outcome* $(\mathbf{X}', \mathbf{s}')$ *for* $\mathsf{IDO}(\mathcal{I})$ *where* $\mathbf{X}'$ *is PROP1, we can compute a PROP outcome* $(\mathbf{X}, \mathbf{s})$ *for instance* $\mathcal{I}$ *in polynomial time, where* $\mathbf{X}$ *is PROP1 and* $\tau(\mathbf{s}) \leq \tau(\mathbf{s}')$*.*

With Lemma 2.6, in the following, we only have to consider instances that are IDO.

3 Allocations of Chores

In this section, we propose an algorithm for computing PROP outcomes for the allocation of chores. Recall that Wu et al. [2023] showed that no algorithm can ensure the existence of PROP outcomes for n agents with a total subsidy smaller than $\alpha(n)$, where

$$\alpha(n) = \begin{cases} n/4 & \text{when } n \text{ is even,} \\ (n^2-1)/(4n) & \text{when } n \text{ is odd.} \end{cases}$$

In the following, we prove the following theorem.

Theorem 3.1. *Given any instance $\mathcal{I} = (M, N, \mathbf{u})$ for the allocation of chores, we can compute in polynomial time a PROP1 allocation $\mathbf{X}$ such that in the corresponding PROP outcome, the total subsidy is at most $\alpha(n)$.*

Our algorithm follows a simple idea:

(1) Suppose that we can partition the items into bundles whose utilities are roughly the same for all agents (e.g., differ by at most 1), then we can treat the n bundles as n "mega-items";

(2) By sequentially allocating each mega-item to the agent requiring the minimum subsidy on it, we can upper bound the total subsidy utilizing the efficiency of subsidization.

Observe that the first step can be easily done for IDO instances in a round-robin manner. For the second step, we can upper bound the total subsidy by $\alpha(n)$ from the perspective of the agent who receives the last bundle. We assume that m is divisible by n by adding dummy items with zero utility to all agents. Following these ideas, we formalize our algorithm as follows.

Algorithm 1: PROP Outcome for Chores

Input: An IDO instance $\mathcal{I} = (M, N, \mathbf{u})$ with $u_i(e_1) \geq u_i(e_2) \geq \cdots \geq u_i(e_m)$ for all $i \in N$.

Let $A_i \leftarrow \{e_j : j = i + z \cdot n \text{ for some integer } z\}$, for all $i \in [n]$;
Let $N' \leftarrow N$;
for $j = 1, 2, \ldots, n$ **do**
  Let $i^* \leftarrow \text{argmin}_{i \in N'} \{\mathsf{PROP}_i - u_i(A_j)\}$;
  Update $X_{i^*} \leftarrow A_j$ and $N' \leftarrow N' \setminus \{i^*\}$;

Output: PROP outcome with allocation $\mathbf{X}$.

As shown in Algorithm 1, we first partition the items into $\mathbf{A} = (A_1, \cdots, A_n)$ in a round-robin manner, and then sequentially allocate each bundle to the active agent who requires the minimum amount of subsidy (to guarantee proportionality), where N' contains all active agents (who have not received a bundle). Clearly, the algorithm runs in polynomial time. Therefore, to prove Theorem 3.1, it suffices to show that the total required subsidy is upper bounded by $\alpha(n)$.

We show that the partition $\mathbf{A}$ is approximately even, i.e., for all agents, the utility difference between any two bundles is small. Moreover, we show that the partition is EF1 for all agents, meaning that as long as each agent receives exactly one bundle, then the allocation is EF1[1].

Lemma 3.2. *For the partition $\mathbf{A}$ computed in Alg. 1, for any agent $i \in N$ and any two bundles A_x, A_y, either $A_x = \emptyset$, or there exists $e \in A_x$ such that $u_i(A_x - e) \geq u_i(A_y)$.*

Proof. Suppose $m = k \cdot n$, we have $|A_x| = |A_y| = k$. By definition we have $A_x = \{e_x, e_{x+n}, e_{x+2n}, \ldots\}$ and $A_y = \{e_y, e_{y+n}, e_{y+2n}, \ldots\}$. Since the instance is IDO, we have

$$u_i(A_x - e_{x+(k-1)n}) = \textstyle\sum_{z=0}^{k-2} u_i(e_{x+zn}) \geq \textstyle\sum_{z=1}^{k-1} u_i(e_{y+zn}) \geq u_i(A_y),$$

which proves the lemma. □

Next, we provide an upper bound for the total subsidy.

Lemma 3.3. *For the allocation $\mathbf{X}$ computed in Alg. 1, the total subsidy of the corresponding PROP outcome $(\mathbf{X}, \mathbf{s})$ is at most $\alpha(n)$.*

Proof. We allocate the bundles in n rounds while in each round $j \in \{1, 2, \ldots, n\}$ we allocate bundle A_j to the active agent $i \in N'$ with minimum $\mathsf{PROP}_i - u_i(A_j)$. For the convenience of notation, we reindex agents and assume w.l.o.g. that agent i receives bundle A_i. Thus, agent n is the last active agent. By definition, when allocating each bundle A_j where $j < n$, agent n is one of the active agents but does not receive bundle A_j. Hence for all $j \in [n]$ we have

$$\mathsf{PROP}_n - u_n(A_j) \geq \mathsf{PROP}_j - u_j(A_j).$$

Therefore, we can upper-bound the total subsidy by

$$\tau(\mathbf{s}) = \textstyle\sum_{j \in N} \max\{\mathsf{PROP}_j - u_j(A_j), 0\} \leq \textstyle\sum_{j \in N} \max\{\mathsf{PROP}_n - u_n(A_j), 0\}.$$

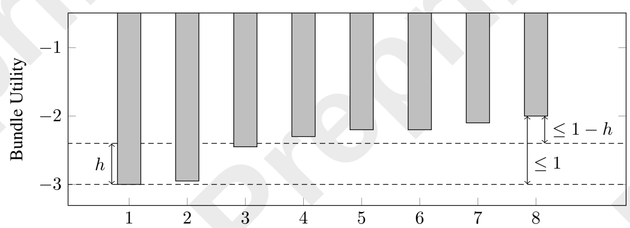

Figure 1: An example for the bundle utilities under agent n's perspective. The difference between any two bundles' utilities is at most 1. The lower horizontal dashed line represents the proportional share of agent n. We have $H = \{A_1, A_2, A_3\}$ and $f = 3$.

Let $H = \{A_j : u_n(A_j) < \mathsf{PROP}_n\}$ be the bundles with utility lower than the proportional share of agent n and $f = |H|$. We use h to denote $\max_{A_j \in H} \{\mathsf{PROP}_n - u_n(A_j)\}$. See Figure 1 for an example. Then we can upper bound the total required subsidy by

$$\tau(\mathbf{s}) \leq \textstyle\sum_{j \in N} \max\{\mathsf{PROP}_n - u_n(A_j), 0\} = \textstyle\sum_{A_j \in H} (\mathsf{PROP}_n - u_n(A_j)) \leq f \cdot h. \quad (1)$$

[1]Note that EF1 holds only for the IDO instances. The IDO reduction cannot guarantee preserving EF1.

On the other hand, observe that $\sum_{j\in N} u_n(A_j) = u_n(M) = n \cdot \mathsf{PROP}_n$, which implies

$$\sum_{A_j \in H} (\mathsf{PROP}_n - u_n(A_j)) = \sum_{A_j \in \mathbf{A}\setminus H} (u_n(A_j) - \mathsf{PROP}_n)\,.$$

Hence, the total subsidy can also be bounded by

$$\tau(\mathbf{s}) \le \sum_{A_j \in \mathbf{A}\setminus H} (u_n(A_j) - \mathsf{PROP}_n) \le (n-f)\cdot(1-h), \quad (2)$$

where the last inequality holds by Lemma 3.2: for all $A_j \in \mathbf{A} \setminus H$ we have

$$u_n(A_j) - \mathsf{PROP}_n \le (\min_{i\in N}\{u_n(A_i)\} + 1) - \mathsf{PROP}_n \le 1 - h.$$

Combining Equations (1) and (2), we upper bound the total subsidy by

$$\tau(\mathbf{s}) = \min\{fh, (n-f)(1-h)\} \le \frac{f(n-f)}{n} \le \alpha(n),$$

where the first inequality holds because when $h \le \frac{n-f}{n}$, we have $fh \le \frac{f(n-f)}{n}$; when $h > \frac{n-f}{n}$, we have $(n-f)(1-h) < \frac{f(n-f)}{n}$. The second inequality holds because $f \in \{1, \ldots, n\}$ and $f(n-f)$ attains the maximum value $\alpha(n)$ when $f = n/2$ for even n; $f = (n-1)/2$ for odd n. □

4 Allocation of Mixed Items

In this section, we show that our algorithm and analysis can be extended to the settings of mixed items. We first consider the mixture of *objective* goods and chores, where every item is either a good to all agents or a chore to all agents. Note that this setting subsumes the allocation of goods and chores as special cases. For this case, we show that the optimal subsidy guarantee $\alpha(n)$ can be obtained. Then we generalize the results to the mixture of *subjective* goods and chores, where an item can be a good to some agent while being a chore to another. We show that every subjective mixed instance can be reduced to an objective mixed instance, and we can transform the allocation for the objective mixed instance to an allocation for the subjective mixed instance, preserving the upper bound on the total subsidy and the PROP1 property.

4.1 Objective Mixed Instance

We first consider the mixture of objective goods and chores [Hosseini *et al.*, 2023b], for which we can partition the items M into M^+ and M^-, where M^+ contains all the goods and M^- contains all the chores. We show that the optimal subsidy guarantee $\alpha(n)$ can be achieved by implementing the double round-robin algorithm, which is proposed by Aziz et al. [2022a] to compute EF1 allocations for the mixed setting. Note that in their setting, the input instance is not necessarily IDO. In the following, we show that for IDO instances, any arbitrary permutation of the bundles in the computed allocation results in an EF1 allocation. We call such a partition an *EF1 partition*.

Definition 4.1 (EF1 Partition)**.** A partition $\mathbf{A}$ is EF1 if for all $i \in N$ and $A_x, A_y \in \mathbf{A}$, either

- $u_i(A_x) \ge u_i(A_y)$; or
- there exists some item $e \in A_x \cup A_y$ such that $u_i(A_x - e) \ge u_i(A_y - e)$.

As we have introduced before, we implement the double round-robin algorithm (see Lines 1 - 7 of Alg. 2) to compute an approximately even partition, which is used to replace the round-robin partition we used in Algorithm 1. Roughly speaking, we allocate the goods and the chores both in a round-robin manner. However, when allocating the goods, bundle A_1 has the highest priority, e.g., e_1 (the best good) is allocated to A_1 while bundle A_n has the lowest priority. When allocating the chores, bundle A_n has the highest priority while bundle A_1 has the lowest priority, e.g., e_m (the worst chore) is allocated to A_1 (see Figure 2 for an illustration). We show in Lemma 4.2 that the computed partition is EF1. We remark that the partition (instead of the allocation) being EF1 is crucial to our setting because we need the freedom to decide the assignment of the bundles to minimize the required total subsidy. Note that we can assume w.l.o.g. that both $|M^+|$ and $|M^-|$ are divisible by n, as we can add dummy items with zero utility to all agents. Suppose $|M^+| = z \cdot n$. Since the instance is IDO, for all $i \in N$ we have

$$u_i(e_1) \ge \cdots \ge u_i(e_{zn}) \ge 0 \ge u_i(e_{zn+1}) \ge \cdots \ge u_i(e_m).$$

Algorithm 2: PROP Outcome for Objective Mixed

Input: An IDO objective mixed instance $\mathcal{I} = (M, N, \mathbf{u})$ with $u_i(e_1) \ge \cdots \ge u_i(e_{zn}) \ge 0 \ge u_i(e_{zn+1}) \ge \cdots \ge u_i(e_m)$ for all $i \in N$.

```
1  Let A_i ← ∅ for all i ∈ N ;
2  for j = 1, 2, ..., zn do
3      Let i ← ((j − 1) mod n) + 1 ;
4      A_i ← A_i + e_j ;
5  for j = zn + 1, ..., m do
6      Let i ← n − ((j − 1) mod n) ;
7      A_i ← A_i + e_j ;
8  Let N' ← N ;
9  for j = 1, 2, ..., n do
10     Let i* ← argmin_{i∈N'} {PROP_i − u_i(A_j)} ;
11     Update X_{i*} ← A_j and N' ← N' \ {i*} ;
```

Output: PROP outcome with allocation $\mathbf{X}$.

Lemma 4.2. *The partition* $\mathbf{A}$ *computed in Alg. 2 is EF1.*

Proof. Fix any agent $i \in N$ and any two bundles $A_x, A_y \in \mathbf{A}$. Suppose $u_i(A_x) < u_i(A_y)$, we show that there exists an item $e \in A_x \cup A_y$ such that $u_i(A_x - e) \ge u_i(A_y - e)$.

We first consider the case when $x < y$. Recall that in the round-robin allocation of goods, bundle A_x has a higher priority and in each round receives an item that is at least as good as the one bundle A_y receives. Therefore we have $u_i(A_x \cap M^+) \ge u_i(A_y \cap M^+)$. On the other hand, for the chores allocated to A_x and A_y, bundle A_y has a higher priority. However, by the property of round-robin, if we

e_m	$\cdots$	e_{m-x+1}	$\cdots$	e_{m-y+1}	$\cdots$	e_{m-n+1}	Chores
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Chores
e_{zn+n}	$\cdots$	$e_{zn+n+1-x}$	$\cdots$	$e_{zn+n+1-y}$	$\cdots$	e_{zn+1}	Chores
e_{zn-n+1}	$\cdots$	e_{zn-n+x}	$\cdots$	e_{zn-n+y}	$\cdots$	e_{zn}	Goods
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Goods
e_1	$\cdots$	e_x	$\cdots$	e_y	$\cdots$	e_n	Goods
A_1	$\cdots$	A_x	$\cdots$	A_y	$\cdots$	A_n	

Figure 2: Illustrations for the partition by double round-robin when $x < y$. When allocating goods, bundle A_x has a higher priority (compared with A_y). On the other hand, bundle A_y has a higher priority when allocating the chores.

remove the last item (the worst chore) e_{m-x+1} that is allocated A_x, the total utility of chores in $A_x - e_{m-x+1}$ is at least that of chores in A_y (see Figure 2 for an illustration). Therefore, combining both goods and chores, we have $u_i(A_x - e_{m-x+1}) \geq u_i(A_y)$.

Next, we consider the case when $x > y$. By a similar argument as above, since A_x has a higher priority than A_y during the allocation of chores, we have $u_i(A_x \cap M^-) \geq u_i(A_y \cap M^-)$. On the other hand, while A_x has a lower priority in receiving goods, by removing the first item (the best good) e_y allocated to A_y, the total utility of goods in A_x is at least that of the goods in $A_y - e_y$. Hence we have $u_i(A_x) \geq u_i(A_y - e_y)$. Since the bundles A_x and A_y and the agent $i \in N$ are chosen arbitrarily, by definition the partition is EF1 for all agents. □

Given that the partition $\mathbf{A}$ is EF1, we can bound the total required subsidy of allocation $\mathbf{X}$ by an analysis almost identical to that of the chores instance (Lemma 3.3).

Lemma 4.3. *For the allocation $\mathbf{X}$ computed in Alg. 2, the total subsidy of the corresponding PROP outcome $(\mathbf{X}, \mathbf{s})$ is at most $\alpha(n)$.*

Combining the above two lemmas with Lemma 2.6, we have proved Theorem 4.4.

Theorem 4.4. *Given any objective mixed instance $\mathcal{I} = (M, N, \mathbf{u})$, we can compute in polynomial time a PROP1 allocation $\mathbf{X}$ such that in the corresponding PROP outcome, the total subsidy is at most $\alpha(n)$.*

Since the allocation of goods is a special case of objective mixed instances, Theorem 4.4 not only generalizes our first result (Theorem 3.1) but also implies that the optimal subsidy guarantee $\alpha(n)$ can be obtained for the allocation of goods.[2]

4.2 Subjective Mixed Instance

In this section, we extend our framework to the setting where an item can be a good for one agent but a chore for another, i.e., the identity of goods and chores can be *subjective*.

Our analysis involves a reduction from the subjective mixed instances to the objective mixed instances[3]. Given any IDO subjective mixed instance $\mathcal{I}$, we construct a new instance $\mathcal{I}'$ which is IDO and objective mixed. By running Algorithm 2 on instance $\mathcal{I}'$, we obtain a PROP outcome $(\mathbf{X}', \mathbf{s}')$ for $\mathcal{I}'$, based on which we construct $(\mathbf{X}, \mathbf{s})$ that is a PROP outcome for the original instance $\mathcal{I}$. We show that the total subsidy $\tau(\mathbf{s}) \leq \tau(\mathbf{s}')$, which implies that the optimal subsidy bound of $\alpha(n)$ can also be achieved for the subjective mixed instance. We first describe our construction of the objective mixed instance $\mathcal{I}$.

Reduction to Objective Mixed Instances Given any subjective mixed instance $\mathcal{I}$ that is IDO, we first identify the agent with the maximum number of goods, i.e.,

$$i^* = \operatorname*{argmax}_{i \in N} |\{e \in M : u_i(e) > 0\}| .$$

Note that we can assume w.l.o.g. that $i^* = 1$. For any agent i, we let k_i be the number of items that are goods to agent i. Therefore we have $k_i \leq k_1$ for all $i \in N$. We construct a new instance $\mathcal{I}' = (M, N, \mathbf{u}')$ as follows: for every agent $i \in N$, we define her utility function u'_i as

$$u'_i(e_j) = \begin{cases} 0, & \text{if } k_i < j \leq k_1 \\ u_i(e_j), & \text{otherwise.} \end{cases}$$

In other words, for the items that are goods to agent 1 but chores to agent i, we raise their utilities to zero in u'_i. Particularly, we have $u'_1 = u_1$. Hence we can view all these items as goods to all agents, resulting in an objective mixed instance.

Observation 1. *The constructed instance $\mathcal{I}'$ is an IDO objective mixed instance.*

Proof. Fix some agent $i \in N$. We show that $u'_i(e_x) \geq u'_i(e_y)$ for any $x < y$. Recall that $\mathcal{I}$ is an IDO instance and $u'_i(e_x) = u_i(e_x)$ holds for any $x \leq k_i$ and $x > k_1$. Hence to show that $\mathcal{I}'$ is IDO, it suffices to consider: 1) $k_i < x < k_1$; and 2) $k_i < y < k_1$.

- If $k_i < x \leq k_1$, we have $u'_i(e_x) = 0$. For any $y > x$, we have $u'_i(e_y) \leq 0 = u'_i(e_x)$.
- If $k_i < y \leq k_1$, we have $u'_i(e_y) = 0$. For any $x < y$, we have $u'_i(e_x) \geq 0 = u'_i(e_y)$.

Hence we have $u'_i(e_1) \geq \cdots \geq u'_i(e_m)$ for all $i \in N$, and the instance is IDO. Moreover, for all agent $i \in N$, we have $u'_i(e_j) \geq 0$ for all $j \leq k_1$ and $u'_i(e_j) \leq 0$ for all $j > k_1$. Therefore the instance $\mathcal{I}'$ is an objective mixed instance. □

[2] It is also shown in [Wu *et al.*, 2023] that no algorithm can guarantee PROP outcomes with subsidy strictly less than $\alpha(n)$ for the allocation of goods.

[3] Similar reductions can be obtained for other share-based fairness notions like maximin share (MMS).

For the constructed instance $\mathcal{I}'$, we use PROP'_i to denote agent i's proportional share, i.e., $\mathsf{PROP}'_i = \frac{u'_i(M)}{n}$. Since the utility of any item on any agent does not decrease during the reduction, we have $\mathsf{PROP}'_i \geq \mathsf{PROP}_i$ for all $i \in N$.

Observation 2. *For all $i \in N$, we have $\mathsf{PROP}'_i \geq \mathsf{PROP}_i$.*

Following Theorem 4.4, we can compute a PROP outcome $(\mathbf{X}', \mathbf{s}')$ for instance $\mathcal{I}'$, where the total subsidy $\tau(\mathbf{s}') \leq \alpha(n)$ and for any agent $i \in N$ we have

$$u'_i(X'_i) + s'_i \geq \mathsf{PROP}'_i.$$

Based on the allocation $\mathbf{X}'$, we construct a new allocation $\mathbf{X}$ for instance $\mathcal{I}$ as follows. For any agent $i \geq 2$, we define S_i as the set containing all the items $e_j \in X'_i$ that are chores to i (under u_i) but goods to agent 1, which can be equivalently expressed as $S_i = \{e_j \in X'_i : k_i < j \leq k_1\}$.

Note that by the construction of u'_i, we have $u'_i(S_i) = 0$ for all $i \geq 2$. Then we define the allocation $\mathbf{X}$ as follows:

- for all $i \geq 2$, let $X_i = X'_i \setminus S_i$;
- for agent 1, let $X_1 = X'_1 \cup \left(\bigcup_{i\geq 2} S_i\right)$.

In other words, we reallocate all items that are chores to its receiver in allocation $\mathbf{X}'$ but goods to agent 1 back to agent 1 in allocation $\mathbf{X}$. We summarize the steps of the whole algorithm in Algorithm 3.

Algorithm 3: PROP Outcome for Subjective Mixed

Input: An IDO subjective mixed instance $\mathcal{I} = (M, N, \mathbf{u})$ with $u_i(e_1) \geq u_i(e_2) \geq \cdots \geq u_i(e_m)$ for all $i \in N$.

```
Let k_i ← max{j ∈ [m] : u_i(e_j) > 0} for all i ∈ N;
Let i* ← argmax_{i∈N} {k_i};
for i ∈ N do
    Let u'_i(e_j) = u_i(e_j) for all j ≤ k_i or j > k_{i*};
    Let u'_i(e_j) = 0 for all k_i < j ≤ k_{i*};
Let (X', s') be the output of Algorithm 2 on instance I' = (M, N, u');
for i ≠ i* do
    Let S_i ← {e_j ∈ X'_i : k_i < j ≤ k_{i*}};
    X_i ← X'_i \ S_i;
X_{i*} ← X'_{i*} ∪ (⋃_{i≠i*} S_i);
```

Output: PROP outcome with allocation $\mathbf{X}$.

We argue that for all $i \in N$, we have $u_i(X_i) \geq u'_i(X'_i)$.

Lemma 4.5. *For any agent $i \in N$, we have $u_i(X_i) \geq u'_i(X'_i)$. Moreover, for all $e \in X_i$ we have $u_i(e) = u'_i(e)$.*

Proof. We first consider agent 1. Since $u_1 = u'_1$ and $X'_1 \subseteq X_1$, it suffices to argue that $u_1(\bigcup_{i\geq 2} S_i) \geq 0$. Note that for any item $e_j \in \bigcup_{i\geq 2} S_i$, we have $j \leq k_1$. Following the definition of k_1, we have $u_i(e_j) > 0$ for all $j \leq k_1$. Hence we have $u_i(\bigcup_{i>2} S_i) > 0$, which implies that $u_i(X_i) > u'_i(X'_i)$.

Next, we consider any agent $i \geq 2$. By the definition of u_i and S_i, we have $u_i(e_j) = u'_i(e_j)$ for all $e_j \in X_i = X'_i \setminus S_i$. Therefore we have

$$\begin{aligned} u_i(X_i) &= u_i(X'_i \setminus S_i) = u'_i(X'_i \setminus S_i) \\ &= u'_i(X'_i) - u'_i(S_i) = u'_i(X'_i), \end{aligned}$$

where the last equality holds since $u'_i(S_i) = 0$. Hence, the lemma follows. □

Finally, we show that $\mathbf{X}$ is a PROP1 allocation, and the corresponding PROP outcome requires a total subsidy $\leq \tau(\mathbf{s}')$.

Lemma 4.6. *The allocation $\mathbf{A}$ computed in Alg. 3 is PROP1.*

Lemma 4.7. *For the allocation $\mathbf{X}$ computed in Alg. 3, the total subsidy of the corresponding PROP outcome $(\mathbf{X}, \mathbf{s})$ is at most $\tau(\mathbf{s}')$.*

Proof. Following Lemma 4.5 and Observation 2, for any agent $i \in N$ we have

$$u_i(X_i) \geq u'_i(X'_i) \geq \mathsf{PROP}'_i - s'_i \geq \mathsf{PROP}_i - s'_i.$$

Since $s_i = \max\{\mathsf{PROP}_i - u_i(X_i), 0\}$, we have $s_i \leq s'_i$ for all $i \in N$. Hence, the total required subsidy is at most $\tau(\mathbf{s}')$. □

Combining the above two lemmas with the result for the objective mixed instances (see Theorem 4.4) and the IDO reduction (see Lemma 2.6), we have proved Theorem 4.8.

Theorem 4.8. *Given any subjective mixed instance $\mathcal{I} = (M, N, \mathbf{u})$, we can compute in polynomial time a PROP1 allocation $\mathbf{X}$ whose corresponding PROP outcome requires subsidy at most $\alpha(n)$.*

5 Conclusion

In this paper, we revisit the problem of fair allocation with subsidy, for goods, chores, and their mixture. We show that even for the mixture of subjective goods and chores, we can compute PROP outcomes with subsidy at most $\alpha(n)$ (which is $n/4$ for even n and $(n^2 - 1)/(4n)$ for odd n). Our upper bound matches the previous lower bound of the required subsidy for PROP allocations in the setting of chores, making it optimal. Unlike previous results based on the rounding framework, we present a new framework that first computes a partition of items into n bundles that are even to all agents and then decides a subsidy-efficient assignment of the bundles.

Our result completes the work of Wu et al. [2023] and generalizes it to the mixed setting. However, there are still many open problems on the topic of fair allocation with subsidy. For example, for the case when agents have general weights, there is still a gap between the upper bound of $n/3$ [Wu and Zhou, 2024] and the lower bound of $n/4$ for the total subsidy. It seems difficult to extend our framework to the weighted setting since when agents have different weights, it is inherently impossible to partition the items into n bundles without deciding their assignments. It would also be interesting to investigate the problem together with efficiency or strategyproofness guarantees. Finally, we believe that it would be a nice topic to study allocations with subsidy for other fairness criteria, e.g., Maximin Share (MMS) or AnyPrice Share (APS). Since these fairness notions are strictly easier to satisfy than proportionality, it would be interesting to know whether a total subsidy strictly less than $n/4$ is sufficient.

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Contribution Statement

The authors are ordered alphabetically.

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