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First-Order Coalition Logic

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Abstract

We introduce *First-Order Coalition Logic* (FOCL), which combines key intuitions behind Coalition Logic (CL) and Strategy Logic (SL). Specifically, FOCL allows for arbitrary quantification over actions of agents. FOCL is interesting for several reasons. First, we show that FOCL is strictly more expressive than existing coalition logics. Second, we provide a sound and complete axiomatisation of FOCL, which, to the best of our knowledge, is *the first axiomatisation* of any variant of SL in the literature. Finally, while discussing the satisfiability problem for FOCL, we reopen the question of the recursive axiomatisability of SL.

1 Introduction

Logics for strategic reasoning constitute a numerous family of formal tools devised to model, verify, and reason about the abilities and strategies of (groups of) autonomous agents in a competitive environment [Pauly, 2002; Alur *et al.*, 2002; van der Hoek *et al.*, 2005; Mogavero *et al.*, 2014; Chatterjee *et al.*, 2010]. Strategies here are 'recipes' telling agents what to do in order to achieve their goals. The competitive environment part arises from the fact that in the presence of several agents trying to achieve their own goals, the actions of one agent may influence the available strategies of another agent. Such logics have been shown to be invaluable for specification and verification within various domains: neuro-symbolic reasoning [Akintunde *et al.*, 2020], voting protocols [Jamroga *et al.*, 2018], autonomous submarines [Ezekiel *et al.*, 2011], manufacturing robots [de Silva *et al.*, 2017], and so on.

The prime representatives of logics for strategic reasoning are *coalition logic* (CL) [Pauly, 2002], *alternating-time temporal logic* (ATL), [Alur *et al.*, 2002], and *strategy logic* (SL) [Mogavero *et al.*, 2010] (and numerous variations thereof). CL extends the language of propositional logic with constructs $\langle\!\langle C \rangle\!\rangle\varphi$ meaning 'coalition C has a joint action such that φ holds in the next state (no matter what agents outside of the coalition do at the same time)'. ATL extends further the abilities of agents to force temporal goals expressed with the help of such modalities as 'Until' and 'Release'. Finally, SL allows for a more fine-tuned quantification over agents' abilities: while in both CL and ATL we have a fixed quantification prefix $\exists\forall$, in SL we can have arbitrary quantification prefixes. Thus, in SL we can reason, for example, about agents sharing their strategies, and such game-theoretic notions like dominant strategies and Nash equilibria. Hence, ATL is strictly more expressive than CL, and, in turn, SL is strictly more expressive than ATL (and its more general cousin ATL*).

Sound and complete axiomatisations of CL [Pauly, 2002; Goranko *et al.*, 2013] and ATL [Goranko and van Drimmelen, 2006] are now classic results in the field. However, to the best of our knowledge, *no axiomatisations of* SL*, nor any of its variants, have been considered in the literature so far*.

In this paper, we introduce a novel variation of the next-time fragment of SL that we call *first-order coalition logic* (FOCL)[1]. As its name suggests, FOCL combines the coalition reasoning capabilities of CL with the first-order features of SL. Specifically, we allow arbitrary quantification prefixes over agents' actions and also allow action labels to appear explicitly in the language. This makes FOCL to be closely related to ATL *with explicit strategies* [Walther *et al.*, 2007].

We first show that FOCL is quite a special CL, being strictly more expressive than other known coalition logics. With such a remarkable expressivity comes the *PSPACE*-complete model checking problem and the undecidable satisfiability problem. While proving the undecidability result, we have also reopened the problem of the recursive axiomatisability of SL, which was until now assumed to be not recursively axiomatisable [Mogavero *et al.*, 2010]. Moreover, we provide a sound and complete axiomatisation of FOCL, which is, as far as we can tell, *the first axiomatisation of any variant of* SL. Thus, we lay the groundwork for the axiomatisations of more expressive fragments and variants of SL.

The rest of the paper is structured as follows: Section 2 defines the syntax and semantics of FOCL, Section 3 examines its expressiveness, Section 4 presents a complete axiomatisation of FOCL, Section 5 addresses complexity, and Section 6 concludes with directions for future work.

2 Syntax and Semantics

Definition 1 (Language)**.** *A* signature *is a triple* $\alpha = \langle n, \mathcal{C}, \mathtt{Ap} \rangle$*, where* $n \geq 1$ *is a natural number,* $\mathcal{C}$ *is a non-empty*

[1]Not to be confused with *quantified coalition logic* [Ågotnes *et al.*, 2008], where quantification is over coalitions and which is as expressive as CL.

countable set of constants*, and* $\mathtt{Ap}$ *is a non-empty countable set of* atomic propositions *(or* atoms*) such that* $\mathtt{Ap} \cap \mathcal{C} = \emptyset$.

Fix a non-empty countable set $\mathtt{V}$ *of* variables *that is disjoint from any other set in any given signature* α*. The* language of first-order coalition logic *(*FOCL*) is defined as*

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\!(t_1, ..., t_n)\!)\, \varphi \mid \forall x \varphi$$

where $p \in \mathtt{Ap}$*,* $t_i \in \mathcal{C} \cup \mathtt{V}$*,* $x \in \mathtt{V}$*, and all the usual abbreviations of propositional logic (such as* $\vee$*,* $\rightarrow$*,* $\leftrightarrow$*) and conventions for deleting parentheses hold. The existential quantifier* $\exists x\varphi$ *is defined as* $\neg\forall x\neg\varphi$*. Formula* $(\!(t_1, ..., t_n)\!)\, \varphi$ *is read as 'after the agents execute actions assigned to* $t_1 \cdots t_n$*,* φ *is true', and* $\forall x\varphi$ *is read as 'for all actions* x*,* φ *holds'. Given a formula* $\varphi \in$ FOCL*, the* size of φ*, denoted by* $|\varphi|$*, is the number of symbols in* φ*.*

Definition 2 (Free Variables)**.** *Given a formula* φ*, we define its* set of free variables $\mathsf{FV}(\varphi)$ *by the following cases:*

1. *If* $\varphi \in \mathtt{Ap}$*, then* $\mathsf{FV}(\varphi) = \emptyset$*;*
2. *If* $\varphi = \neg\varphi_1$*, then* $\mathsf{FV}(\varphi) = \mathsf{FV}(\varphi_1)$*;*
3. *If* $\varphi = \varphi_1 \wedge \varphi_2$*, then* $\mathsf{FV}(\varphi) = \mathsf{FV}(\varphi_1) \cup \mathsf{FV}(\varphi_2)$*;*
4. *if* $\varphi = (\!(t_1, ..., t_n)\!)\, \varphi_1$*, then* $\mathsf{FV}(\varphi) = \mathsf{FV}(\varphi_1) \cup \{t_i \mid t_i \in \mathtt{V}\}$*;*
5. *if* $\varphi = \forall x\varphi_1$*, then* $\mathsf{FV}(\varphi) = \mathsf{FV}(\varphi_1) \setminus \{x\}$*.*

A formula φ *such that* $\mathsf{FV}(\varphi) = \emptyset$ *is called a* closed formula*, or a* sentence*.*

Definition 3 (Kripke Frame)**.** *A* Kripke frame *is a tuple* $\mathcal{F} = \langle \Sigma, S, R \rangle$*, where* Σ *is a non-empty countable alphabet,* S *is a non-empty set of states s.t.* $\Sigma \cap S = \emptyset$*, and* $R \subseteq S \times \Sigma \times S$ *is a ternary relation, dubbed* transition relation*.* $\mathcal{F}$ *is* serial *if for every* $s \in S$ *and* $a \in \Sigma$*, there is a* $t \in S$ *s.t.* $\langle s, a, t \rangle \in R$*.* $\mathcal{F}$ *is* functional *whenever for all* $s, t, v \in S$ *and for every* $a \in \Sigma$*, if* $\langle s, a, t \rangle \in R$ *and* $\langle s, a, v \rangle \in R$*, then* $t = v$*.*

Definition 4 (Concurrent Game Structure)**.** *A* game frame *is a tuple* $\mathcal{G} = \langle n, \mathtt{Ac}, \mathcal{D}, S, R \rangle$ *with triple* $\langle \mathcal{D}, S, R \rangle$ *being a serial and functional Kripke frame, where:* n *is a positive natural number and* $\mathcal{D}$ *is a set of tuples of elements of* $\mathtt{Ac}$ *of length* n *(elements of this set will be called* decisions*).*

A Concurrent Game Structure *(CGS) is a pair* $\mathfrak{G} = \langle \mathcal{G}, \mathcal{V} \rangle$*, where* $\mathcal{G}$ *is a game frame, and* $\mathcal{V} : \mathtt{Ap} \rightarrow \mathcal{P}(S)$ *is a* valuation function *assigning to each atomic proposition a subset of* S*.*

Let $Prop(s) = \{p \in \mathtt{Ap} \mid s \in \mathcal{V}(p)\}$ *be the set of all atomic propositions true in state* s*. We define the* size of CGS $\mathfrak{G}$ *as* $|\mathfrak{G}| = n + |\mathtt{Ac}| + |\mathcal{D}| + |S| + |R| + \sum_{s \in S} |Prop(s)|$*, where* $|\mathcal{D}| = |\mathtt{Ac}|^n$*. We call CGS* $\mathfrak{G}$ finite*, if* $|\mathfrak{G}|$ *is finite.*

Definition 5. *Given a signature* $\alpha = \langle m, \mathcal{C}, \mathtt{Ap} \rangle$*, and a CGS* $\mathfrak{G} = \langle n, \mathtt{Ac}, \mathcal{D}, S, R, \mathcal{V} \rangle$*, we say that* $\mathfrak{G}$ *is constructed over* α *iff* $m = n$ *and* $\mathcal{C} = \mathtt{Ac}$*.*

Definition 6 (Satisfaction)**.** *Let* φ *be a sentence and* $\mathfrak{G}$ *be a CGS that are both constructed over the same signature* α*. The* satisfaction relation $\mathfrak{G}, s \models \varphi$ *is inductively defined as follows:*

$\mathfrak{G}, s \models p$	*iff* $s \in \mathcal{V}(p)$
$\mathfrak{G}, s \models \neg\psi$	*iff* $\mathfrak{G}, s \not\models \psi$
$\mathfrak{G}, s \models \psi \wedge \chi$	*iff* $\mathfrak{G}, s \models \psi$ *and* $\mathfrak{G}, s \models \chi$
$\mathfrak{G}, s \models (\!(a_1, ..., a_n)\!)\, \psi$	*iff* $\exists t \in S$ *s.t.* $\langle s, a_1, ..., a_n, t \rangle \in R$ *and* $\mathfrak{G}, t \models \psi$
$\mathfrak{G}, s \models \forall x\psi$	*iff* $\forall a \in \mathtt{Ac} : \mathfrak{G}, s \models \psi[a/x]$

where $a_1, ..., a_n$ *are constants, and* $\psi[a/x]$ *denotes the result of substituting every occurrence of the variable* x *with the constant* a *in* ψ*. We will also sometimes write* $\vec{a}$ *for* $a_1 \cdots a_n$*.*

Definition 7 (Closure of a Formula)**.** *Given a formula* φ *whose set of free variables is* $\{x_1, \ldots, x_n\}$*, we denote by* $C(\varphi)$ *the* closure *of* φ*, which is the formula* $\forall x_1 \cdots \forall x_n \varphi$*.*

Definition 8 (Validity)**.** *Let* $\mathfrak{G}$ *be a CGS constructed over a signature* α*, and* φ *a formula constructed over* α*. Given a state* s *of* $\mathfrak{G}$*, we write* $\mathfrak{G}, s \models \varphi$ *iff* $\mathfrak{G}, s \models C(\varphi)$*. We say that* φ *is* valid in a CGS $\mathfrak{G}$ *(written* $\mathfrak{G} \models \varphi$*) iff* $\mathfrak{G}, s \models \varphi$ *for every state* s *of* $\mathfrak{G}$*. Finally, we say that* φ *is* valid *(written* $\models \varphi$*) iff it is valid in every CGS constructed over a signature with* n *agents. Given a set of formulae* X*, we write* $\mathfrak{G}, s \models X$ *if for every formula* $\varphi \in X$*,* $\mathfrak{G}, s \models \varphi$*. Finally, we write* $X \models \psi$ *and we say that* ψ *is a logical consequence of* X *iff* $\mathfrak{G} \models X$ *implies* $\mathfrak{G} \models \psi$ *for every CGS* $\mathfrak{G}$ *constructed over the same signature as* ψ *and formulae in* X*.*

Remark 1. *Note that the truth of open (i.e. not closed) formulae is reduced to the truth of the closed ones via closure (Definition 7). This approach is fairly standard in first-order logic (see, e.g., [van Dalen, 1994]). We could also define the truth of a formula w.r.t. an assignment, but this would not affect the results presented here. Our choice simplifies the formal machinery of the paper and makes it more readable.*

The next proposition, that follows straightforwardly from the seriality and functionality of frames, shows that we can give an alternative and equivalent characterisation of the truth of a strategic formula in a state of a CGS.

Proposition 1. *Let* $\mathfrak{G} = \langle n, \mathtt{Ac}, \mathcal{D}, S, R, \mathcal{V} \rangle$ *be a CGS,* $s \in S$*, and* $\varphi = (\!(a_1, ..., a_n)\!)\, \psi$*, and suppose that both* $\mathfrak{G}$ *and* φ *are constructed over the same signature* α*. Then* $\mathfrak{G}, s \models \varphi$ *iff* $\forall t \in S$*:* $\langle s, a_1, \ldots, a_n, t \rangle \in R$ *implies* $\mathfrak{G}, t \models \psi$*.*

Remark 2. *Due to Proposition 1, we we have that* $\mathfrak{G}, s \models (\!(\vec{a})\!)\, \psi$ *iff* $\mathfrak{G}, t \models \psi$ *for the unique* t *such that* $\langle s, \vec{a}, t \rangle \in R$*.*

Thus, FOCL can be also viewed as an extension of multi-modal logic [Blackburn *et al.*, 2001; Hennessy and Milner, 1980] for serial and functional frames with first-order quantification over components of arrow labels (i.e. actions).

Example 1. *As observed in [Belardinelli* et al.*, 2019], strategy logics are expressive enough to capture Stackelberg equilibrium (SE). Such an equilibrium is applicable to scenarios where a leader commits to a strategy, and the follower, observing the strategy of the leader, provides her best response. SE is prominent in security games [Sinha* et al.*, 2018], where the attacker observes the defender committing to a defensive strategy and then decides on the best way to attack (if at all).*

We can express such a scenario for the case of one-step strategies by the FOCL *formula* $\forall x_d \exists x_a \forall x_e (\!(x_d, x_a, x_e)\!)\, win_a$*, which intuitively means that for all actions of the defender, the attacker has a counter-action guaranteeing the win for all actions of the environment.*

Similarly to [Mogavero et al.*, 2010], with* FOCL *we can express the existence of deterministic Nash equilibrium (NE) for Boolean goals. If* $\psi_1, ..., \psi_n$ *are goal formulae of agents, we can assert the existence of strategy profile* $x_1, ..., x_n$ *such that if any agent* i *achieves her goal* ψ_i *by deviating from* $x_1, ..., x_n$*, then she can also achieve her goal by sticking to the action profile. The existence of such a profile can be expressed by the following* FOCL *formula:*

$$\exists x_1, ..., x_n (\bigwedge_{i=1}^{n} \exists y_i (\!(x_1, ..., y_i, ..., x_n)\!)\, \psi_i \to \to (\!(x_1, ..., x_i, ..., x_n)\!)\, \psi_i)$$

FOCL *also allows for strategy sharing. Consider examples of CGSs presented in Figure 1. In structure* $\mathfrak{G}_1$ *we have two states* s *and* t*, and the agents can transition between the two states if they synchronise on their actions, i.e. execute the same actions. It is easy to verify that* $\mathfrak{G}_1, s \models \forall x \exists y (\!(x, y)\!)\, p$ *and* $\mathfrak{G}_1, s \models \forall x (\!(x, x)\!)\, \neg p$.

The fact that FOCL is able to capture the Stackelberg and Nash equilibria is significant, since, compared to SL and its fragments that can also capture *both* equilibria, the complexity of the model checking problem for FOCL is *PSPACE*-complete as shown in the proof of Theorem 2 (compared to the range from *2ExpTime* to non-elementary for various SL's [Mogavero *et al.*, 2014]). Moreover, the ability to capture the equilibria can have a significant impact on the prospective applications of FOCL. In particular, it was argued [van der Meyden, 2019; Galimullin and Ågotnes, 2022] that CL is suitable for specification and verification of *atomic swap* smart contracts that allow agents to exchange their assets or private information, like passwords, on a blockchain without necessarily trusting each other. We can use FOCL to verify that acting honestly is indeed a NE for a given specification of a contract. Moreover, having the ability to express strategy sharing, we can verify that a swap is still executable in the situation, where a malicious agent that gained access to the communication channel poses as one of the honest ones by executing the same actions[2].

3 Relation to Other Coalition Logics

In order to appreciate the richness of FOCL, we compare the logic to other CL's. In our comparison we can use two salient features of FOCL. First, the logic allows for *arbitrary quantification prefixes for agents' actions*. This includes using the same strategy variable for different agents to capture *strategy sharing*. The second special feature of FOCL is the presence of *explicit action labels* in its syntax.

Definition 9 (Expressivity)**.** *Let* L_1 *and* L_2 *be two languages, and let* $\varphi \in \mathsf{L}_1$ *and* $\psi \in \mathsf{L}_2$*. We call* φ *and* ψ equivalent*, if for any CGS* $\mathfrak{G}$ *and state* $s \in \mathfrak{G}$*:* $\mathfrak{G}, s \models \varphi$ *iff* $\mathfrak{G}, s \models \psi$*. If for all* $\varphi \in \mathsf{L}_1$ *there exists an equivalent* $\psi \in \mathsf{L}_2$*, then* L_2 *is* at least as expressive as L_1 *(*$\mathsf{L}_1 \leqslant \mathsf{L}_2$*). And* L_2 *is* strictly more expressive than L_1 *(*$\mathsf{L}_1 < \mathsf{L}_2$*) if* $\mathsf{L}_1 \leqslant \mathsf{L}_2$ *and* $\mathsf{L}_2 \nleqslant \mathsf{L}_1$*.*

These two features of FOCL, arbitrary quantification prefixes and explicit actions, on their own are not unique in the landscape of logics for strategic reasoning. Arbitrary quantification prefixes are a hallmark feature of the whole family of *strategy logics* (see, e.g., [Mogavero *et al.*, 2010; Belardinelli *et al.*, 2019]), to which FOCL belongs. Indeed, FOCL can be considered as a variation of the next-time fragment of SL. The idea to refer to actions in the language has also been explored, with a prime example being ATL *with explicit strategies* (ATLES) [Walther *et al.*, 2007]. Another example of such a logic is *action logic* [Borgo, 2007].

Even though both of the main features of FOCL have been explored in the literature, to our knowledge, FOCL is the first logic for strategic reasoning that combines *both* of them.

Coalition logic and quantified coalition logic The original *coalition logic* (CL) [Pauly, 2002], similarly to ATL, allows only single alternation of quantifiers in coalitional modalities. Moreover, this quantification is implicit. Thus, CL extends the language of propositional logic with constructs $\langle\!\langle C \rangle\!\rangle \varphi$ that mean 'there is a strategy for coalition C to achieve φ in the next step'. In *quantified coalition logic* (QCL) [Ågotnes *et al.*, 2008], constructs $\langle\!\langle C \rangle\!\rangle \varphi$ are substituted with $\langle P \rangle \varphi$ meaning 'there exists a coalition C satisfying property P such that C can achieve φ'. Since QCL is as expressive as CL (although exponentially more succinct), we will focus only on CL.

To introduce the semantics of CL, we will denote the choice of actions by coalition $C \subseteq Agt$ with $|Agt| = n$ as σ_C, and denote $Agt \setminus C$ as $\overline{C}$. Finally, $\sigma_C \cup \sigma_{\overline{C}} \in \mathcal{D}$ is a decision. The semantics of $\langle\!\langle C \rangle\!\rangle \varphi$ for a given CGS $\mathfrak{G}$ is then defined as

$$\mathfrak{G}, s \models \langle\!\langle C \rangle\!\rangle \varphi \text{ iff } \exists \sigma_C, \forall \sigma_{\overline{C}} : \mathfrak{G}, t \models \varphi \quad \text{with } t \in S \text{ s.t. } \langle s, \sigma_C \cup \sigma_{\overline{C}}, t \rangle \in R.$$

The translation from formulas CL to formulas of FOCL can be done recursively using the following schema for coalitional modalities: $tr(\langle\!\langle C \rangle\!\rangle \varphi) \to \exists \vec{x} \forall \vec{y} (\!(\vec{x}, \vec{y})\!)\, tr(\varphi)$, where variables $\vec{x}$ (all different) quantify over actions of C, and $\vec{y}$ (all different) quantify over actions of $Agt \setminus C$.

At the same time, one cannot refer to particular actions in CL formulas, as well as express sharing strategies between agents. We can exploit either of these features to show that FOCL is strictly more expressive than CL. Indeed, consider a FOCL formula $\exists x (\!(x, x)\!)\, \neg p$ meaning that there is an action that *both* agents 1 and 2 should use to reach a $\neg p$-state. We can construct two CGSs that are indistinguishable by any CL formulas. At the same time, $\exists x (\!(x, x)\!)\, \neg p$ will hold in one structure and be false in another.

Consider two structures depicted in Figure 1. It is easy to see that $\mathfrak{G}_1, s$ and $\mathfrak{G}_2, s$ cannot be distinguished by any CL formula[3]. Indeed, both structures agree on the valuation of

[2] This is the classic person-in-the-middle attack scenario in cryptography.

[3] These structures are, in fact, in the relation of *alternating bisimulation* [Ågotnes *et al.*, 2007], and hence satisfy the same formulas of CL and ATL. The discussion of bisimulations for all the logics we mention is, however, beyond the scope of this paper, and we leave it for future work.

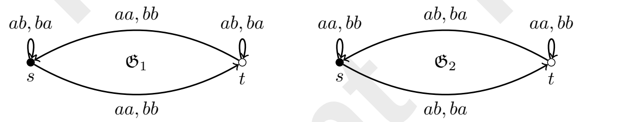

Figure 1: CGSs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ for two agents and two actions. Propositional variable p is true in black states.

propositional variable p in corresponding states. Moreover, none of the agents, 1 and 2, can on their own force a transition from state s to state t. At the same time, the grand coalition $\{1, 2\}$ can match any transition in one structure with a transition with the same effect in the other structure. Now, we can verify that $\mathfrak{G}_1, s \models \exists x(\!(x, x)\!) \neg p$ and $\mathfrak{G}_2, s \not\models \exists x(\!(x, x)\!) \neg p$. For the case of $\mathfrak{G}_1, s \models \exists x(\!(x, x)\!) \neg p$, it is enough to assign action a to x to have $\mathfrak{G}_1, s \models (\!(a, a)\!) \neg p$. To make $\exists x(\!(x, x)\!) \neg p$ hold in $\mathfrak{G}_2, s$, one needs to provide an action that once executed by both agents will force the transition to state t. It is easy to see that there is no such an action in $\mathfrak{G}_2, s$.

Having the translation from CL to FOCL on the one hand, and the indistinguishability result on the other, we hence conclude that FOCL is *strictly more expressive* than CL.

Proposition 2. CL $<$ FOCL.

Conditional strategic reasoning and socially friendly CL With the expressive power of FOCL we can go much further than the classic CL. In particular, we can express in our logic such interesting CL's like *logic for conditional strategic reasoning* (ConStR) [Goranko and Ju, 2022], *socially friendly* CL (SFCL) [Goranko and Enqvist, 2018], and *group protecting* CL (GPCL) [Goranko and Enqvist, 2018].

Presenting the semantics of the aforementioned logic is beyond the scope of this paper. However, we would like to point out that all of the logics can be captured by *basic strategy logic* (BSL) [Goranko, 2023], a variant of SL, where each agent has her own associated strategy variable. Differently from FOCL, BSL allows for all standard temporal modalities like 'neXt', 'Until' and 'Globally'. At the same time, BSL does not allow for variable sharing and does not explicitly refer to actions or strategies. Moreover, it is conjectured that BSL does not have a recursive axiomatisation, while FOCL has a finitary complete axiomatisation (see Section 4).

Translations of all coalition logics introduced in this paragraph into formulas of BSL are presented in [Goranko, 2023], where it is also claimed that BSL is strictly more expressive than all the aforementioned logics. The translation does not employ any temporal features of BSL apart from 'neXt', and thus the same translation also works for FOCL. Moreover, we can use either strategy sharing or explicit actions to argue that FOCL is strictly more expressive than the considered coalition logics. As an example, an argument for the case of SFCL is given in [Catta *et al.*, 2025].

Proposition 3. ConStR $<$ FOCL, SFCL $<$ FOCL, GPCL $<$ FOCL.

Action logic A perhaps most relevant to FOCL coalition logic in the literature is *action logic* (AL) [Borgo, 2007], which is a fragment of *multi-agent PDL with quantificaiton* (mPDLQ) [Borgo, 2005b]. AL extends the language of propositional logic with so-called *modality markers* $[M]$, which are, essentially, prefixes of size $|Agt| = n$, each element of which can either be a quantifier $Q_i x_i$ with $Q_i \in \{\forall, \exists\}$ or an explicit action [Borgo, 2005b; Borgo, 2005a]. An important feature here is that *there are no repeating variables in modality markers*. Finally, to the best of our knowledge, there is no axiomatisation of AL.

Given a modality marker $[M]$, we denote by $\sigma_\exists$ a choice by all the existentially quantified agents, by $\sigma_\forall$ a choice by all the universally quantified agents, and by σ_{act} explicit actions in the corresponding positions in $[M]$. Then modality markers have the following semantics:

$$\mathfrak{G}, s \models [M]\varphi \text{ iff } \exists \sigma_\exists, \forall \sigma_\forall : \mathfrak{G}, t \models \varphi$$
$$\text{with } t \in S \text{ s.t. } \langle s, \sigma_\exists \cup \sigma_\forall \cup \sigma_{act}, t\rangle \in R.$$

Intuitively, $\mathfrak{G}, s \models [M]\varphi$ holds if and only if there is an assignment of actions to all existentially quantified variables in modality marker M such that no matter which actions are assigned to the universally quantified variables, once combined with the explicit actions, the outcome state satisfies φ. This is in line with the semantics of CL as we basically choose actions for a coalition (existentially quantified variables) and verify ψ in all possible outcomes given this choice.

Formulae $[M]\varphi$ of AL can be translated into formulae of FOCL of the form $\exists \vec{x} \forall \vec{y} (\!(t_1, ..., t_n)\!)\, \varphi$, where $\vec{x}$ and $\vec{y}$ with $|\vec{x}| + |\vec{y}| \leqslant n$ are possibly empty sequences of variables for the existentially and universally quantified agents respectively, $t_i := x_i$ if there is a quantifier in position i in the modality marker, and $t_i := a_i$ if there is action a_i in the ith posiiton in the modality marker. Also recall that AL does not allow for sharing strategies (while FOCL does), i.e. all $x_1, ..., x_m$ in the modality marker are unique.

To show that FOCL is more expressive than AL, we need the following proposition. Its proof utilises the strategy sharing feature of FOCL and can be found in [Catta *et al.*, 2025].

Proposition 4. AL *is not at least as expressive as* FOCL.

Having the translation from AL to FOCL on the one hand, and Lemma 4 on the other, we can conclude that FOCL is strictly more expressive than AL.

Corollary 1. AL $<$ FOCL.

The expressivity landscape In this section we have explored the relationship between FOCL and other notable CL's from the literature. The overall expressivity landscape of the considered logics is presented in Figure 2.

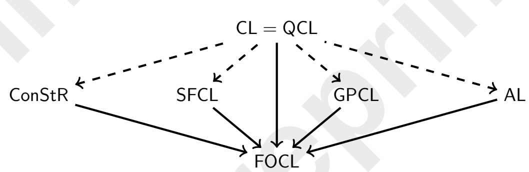

Figure 2: Overview of the expressivity results. An arrow from L_1 to L_2 means $L_1 < L_2$. Dashed arrows represent results from the literature. Solid arrows are new results.

4 Proof Theory

Perhaps the best-known results in the field are complete axiomatisations of CL [Pauly, 2002; Goranko *et al.*, 2013] and ATL [Goranko and van Drimmelen, 2006] (see [Walther *et al.*, 2006; Goranko and Shkatov, 2009] for more constructive approaches). Other completeness results include axiomatisations for logics based on CL and ATL, like already mentioned SFCL [Goranko and Enqvist, 2018], ATLES [Walther *et al.*, 2007], as well as *epistemic* CL [Ågotnes and Alechina, 2019], *resource-bounded* CL [Alechina *et al.*, 2011] and ATL [Nguyen *et al.*, 2018], and ATL *with finitely bounded semantics* [Goranko *et al.*, 2019], to name a few.

In the context of strategy logics, we have quite an opposite picture. Since the inception of SL [Mogavero *et al.*, 2010], its axiomatisation has been an open problem. The same can be said about any of the fragments of SL. The lack of axiomatisations of *any* (fragment of) SL can be traced back to the two main features of the logic: quantification over strategies and arbitrary quantification prefixes.

Indeed, arbitrary alternation of quantifiers in SL is quite different from the fixed quantification prefix of CL and ATL that allow only prefixes $\exists\forall$ and $\forall\exists$. Secondly, quantification over strategies[4] in SL is essentially a second-order quantification over functions. We believe that these two features combined are the root cause of the fact that no complete axiomatisations of (fragments of) SL have been proposed so far.

In FOCL we focus on arbitrary quantification prefixes. To solve this sub-problem, we consider only neXt-time modalities $(\!(t_1 \cdots t_n)\!)\,\varphi$ and deal with the immediate outcomes of agents' choices. This allows us, in particular, to consider quantification over actions rather than strategies. Hence, quantification in FOCL is a first-order quantification, instead of the second-order quantification of SL.

In our proof, we take as inspiration the completeness proof for *first-order modal logic* (FOML) with constant domains [Garson, 1984]. Our construction is quite different, though, as in FOML variables appear in n-ary predicates, and in FOCL variables are placeholders for transition labels.

4.1 Axiomatisation of FOCL

Definition 10 (Axiomatisation)**.** *The* axiom system *for* FOCL *consists of the following axiom schemata and rules, where* $\vec{t} = t_1, \ldots, t_n$ *for* $n \geqslant 1$*, and* t *and each* t_i *are either a variable or a constant.*

PC	*Every propositional tautology*
K	$((\!(\vec{t})\!)\,\varphi \wedge (\!(\vec{t})\!)\,\psi) \leftrightarrow (\!(\vec{t})\!)\,(\varphi \wedge \psi)$
N	$\neg(\!(\vec{t})\!)\,\varphi \leftrightarrow (\!(\vec{t})\!)\,\neg\varphi$
E	$\forall x\varphi \to \varphi[t/x]$
B	$\forall x(\!(\vec{t})\!)\,\varphi \to (\!(\vec{t})\!)\,\forall x\varphi$, *s.t.* $t_i \neq x$ *for all* t_i
MP	*From* $\varphi, \varphi \to \psi$*, infer* ψ
Nec	*From* φ*, infer* $(\!(\vec{t})\!)\,\varphi$
Gen	*From* $\varphi \to \psi[t/x]$*, infer* $\varphi \to \forall x\psi$*, if* $t \notin \varphi$

An axiomatic derivation π *is a finite sequence of formulae* $\varphi_1, \ldots, \varphi_m$ *where for each* $i \leqslant m$*: either* φ_i *is an instance of one of the axiom schemata of* FOCL*, or it is obtained from some preceding formulae in the sequence using rules* MP*,* Nec*, or* Gen*. We write* $\vdash \varphi$ *and say that* φ *is* FOCL derivable *(or simply derivable) iff there is a derivation* π *whose last element is* φ*. Given a set of formulae* X*, we write* $X \vdash \varphi$ *iff there is a finite subset* Y *of* X *such that* $\vdash \bigwedge Y \to \varphi$*.*

[4] A *(memoryless) strategy* for an agent $i \in n$ is a function $\sigma_i : S \to \texttt{Ac}$.

We will freely use the following proposition in the rest of the paper. Its proof is standard, and we omit it for brevity.

Proposition 5. *The following formulae are* FOCL *derivable, where* $\vec{t} = t_1, \ldots, t_n$*, and each* t_i *is either a constant or a variable:*

1. $(\!(\vec{t})\!)\,(\varphi \to \psi) \to ((\!(\vec{t})\!)\,\varphi) \to ((\!(\vec{t})\!)\,\psi)$;
2. $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$ *with* $x \notin \mathsf{FV}(\varphi)$;
3. $\exists z(\varphi \to \forall y\varphi)$ *with* $z \notin \mathsf{FV}(\forall y\varphi)$.

Moreover, if $\varphi \to \psi$ *is derivable, so is* $(\!(\vec{t})\!)\,\varphi \to (\!(\vec{t})\!)\,\psi$*.*

Lemma 1. *Each axiom schema of* FOCL *is valid and each rule of* FOCL *preserves validity.*

The proof of Lemma 1 is done by the application of the definition of the semantics, and it can found in [Catta *et al.*, 2025].

Our completeness proof is based on the canonical model construction, where states are maximal consistent sets with the $\forall$-property.

Definition 11 (Maximal Consistent Sets)**.** *Let* Z *be a set of* FOCL *sentences over a given signature and* $X \subseteq Z$*. We say that: (i)*X *is* consistent *iff* $X \nvdash \bot$*, (ii)* X *is* maximally consistent *(MCS) iff it is consistent and there is no other consistent set of sentences* $Y \subseteq Z$ *s.t.* $X \subset Y$*, and (iii)* X *has the* $\forall$-property *iff for every formula* φ *over the same signature as* Y *and variable* x*, there is a constant* a *such that* $\varphi[a/x] \to \forall x\varphi \in X$*, where* $\varphi[a/x]$ *is closed. We will call a set satisfying all the three requirements* $\forall$*-MCS.*

Let $\alpha = \langle n, \mathcal{C}, \texttt{Ap}\rangle$ be a signature. We denote by $\alpha^\star$ the signature $\langle n, \mathcal{C} \cup \mathcal{C}^\star, \texttt{Ap}\rangle$ where $\mathcal{C}^\star$ is countably infinite, and $\mathcal{C} \cap \mathcal{C}^\star = \emptyset$.

Next lemma shows that each consistent set of sentences over a given signature α can be extended to a consistent set of sentences over $\alpha^\star$ having the $\forall$-property. Its proof follows the standard technique in FOML [Cresswell and Hughes, 1996], and can be found in [Catta *et al.*, 2025].

Lemma 2. *If* X *is a consistent set of sentences over a given signature* α*, then there is a consistent set of sentences* Y *over* $\alpha^\star$ *such that* $X \subseteq Y$*, and* Y *has the* $\forall$*-property.*

The proof of the following lemma (Lindenbaum Lemma) is standard, and we omit it for brevity.

Lemma 3. *Let* X *be a consistent set of sentences over a given signature, then there is an MCS* Y *over the same signature such that* $X \subseteq Y$*.*

The next two lemmas will be instrumental in the proof of the Truth Lemma, and showing that the canonical model we are to define in this proof is indeed a CGS.

Lemma 4. *Let* X *be a consistent set of sentences over a given signature and let* $\vec{a}$ *be a tuple of constants, then the set* $Y_{\vec{a}} = \{\psi \mid (\!(\vec{a})\!)\,\psi \in X\}$ *is also consistent.*

Proof. If $Y_{\vec{a}}$ is empty, the result is trivially valid. Assume that $\varphi \in Y_{\vec{a}}$ and suppose towards a contradiction, that set $Y_{\vec{a}}$ is not consistent. This implies that $(\psi_1 \wedge \cdots \wedge \psi_m) \rightarrow \neg\varphi$ for some finitelty many $\psi_1, \ldots, \psi_m$ in $Y_{\vec{a}}$. Using Proposition 5(1) and propositional reasoning, we can then derive $(\!(\vec{a})\!)\, \psi_1 \wedge \cdots \wedge (\!(\vec{a})\!)\, \psi_m \rightarrow (\!(\vec{a})\!)\, \neg\varphi$. Since $(\!(\vec{a})\!)\, \psi_i \in X$, we conclude by MP that $X \vdash (\!(\vec{a})\!)\, \neg\varphi$. Then, by N and MP we can further derive $X \vdash \neg(\!(\vec{a})\!)\, \varphi$, which contradicts $(\!(\vec{a})\!)\, \varphi \in X$. □

Lemma 5. *Let X be a $\forall$-MCS over a given signature containing infinitely many constants. Then there exists a $\forall$-MCS Y over the same signature such that $Z = \{\psi \mid (\!(\vec{a})\!)\, \psi \in X\} \subseteq Y$.*

Proof. If $Z = \emptyset$, the lemma trivially holds. So assume $\varphi \in Z$, and let E be an enumeration of all sentences of the form $\forall x\xi$, and C an enumeration of the constants in the given signature. We define a sequence of sentences $\theta_0, \theta_1, \ldots$ where $\theta_0 = \varphi$, and given θ_n, we set $\theta_{n+1} = \theta_n \wedge (\xi[a/x] \rightarrow \forall x\xi)$, where $\forall x\xi$ is the $(n+1)$-th formula in E, and a is the first constant in C such that

$$(\star) \quad Z \cup \{\theta_n \wedge (\xi[a/x] \rightarrow \forall x\xi)\} \text{ is consistent.}$$

Let $Y = Z \cup \{\theta_n \mid n \in \mathbb{N}\}$. Clearly Y has the $\forall$-property, and it is consistent if $Z \cup \{\theta_n\}$ is consistent for every $n \in \mathbb{N}$. To show this, we prove that if $Z \cup \{\theta_n\}$ is consistent, then there always exists a constant a satisfying $(\star)$. The set $Z \cup \{\theta_0\} = Z$ is consistent by Lemma 4.

Suppose, towards a contradiction, that $Z \cup \{\theta_n\}$ is consistent but for every constant a, the set $Z \cup \{\theta_n \wedge (\xi[a/x] \rightarrow \forall x\xi)\}$ is inconsistent. Then for each constant a, there exist finitely many formulas $\psi_1^a, \ldots, \psi_m^a$ in Z such that $(\psi_1^a \wedge \cdots \wedge \psi_m^a) \rightarrow (\theta_n \rightarrow \neg(\xi[a/x] \rightarrow \forall x\xi))$ is derivable. From this, by the rules of FOCL, it follows that $((\!(\vec{a})\!)\, \psi_1^a \wedge \cdots \wedge (\!(\vec{a})\!)\, \psi_m^a) \rightarrow (\!(\vec{a})\!)\, (\theta_n \rightarrow \neg(\xi[a/x] \rightarrow \forall x\xi))$ is derivable. Since $\psi_i^a \in Z$ implies $(\!(\vec{a})\!)\, \psi_i^a \in X$, we conclude that (i) $(\!(\vec{a})\!)\, (\theta_n \rightarrow \neg(\xi[a/x] \rightarrow \forall x\xi)) \in X$ for every constant a.

Let z be a variable that occurs neither in θ_n nor in ξ. Consider the sentence $\forall z\, (\!(\vec{a})\!)\, (\theta_n \rightarrow \neg(\xi[z/x] \rightarrow \forall x\xi))$. From the $\forall$-property of X and (i), it follows that $\forall z\, (\!(\vec{a})\!)\, (\theta_n \rightarrow \neg(\xi[z/x] \rightarrow \forall x\xi)) \in X$. By axiom B, this implies $(\!(\vec{a})\!)\, \forall z(\theta_n \rightarrow \neg(\xi[z/x] \rightarrow \forall x\xi)) \in X$, and thus, by (2) of Prop. 5, we have (ii) $(\!(\vec{a})\!)\, (\theta_n \rightarrow \forall z\neg(\xi[z/x] \rightarrow \forall x\xi)) \in X$. Since $\exists z(\xi[z/x] \rightarrow \forall x\xi)$ is derivable in FOCL, applying rule Nec gives $(\!(\vec{a})\!)\, \exists z(\xi[z/x] \rightarrow \forall x\xi) \in X$. From this, together with (ii), and using Proposition 5, we conclude $(\!(\vec{a})\!)\, \neg\theta_n \in X$. By the construction of Z, this implies $\neg\theta_n \in Z$, which contradicts the assumption that $Z \cup \{\theta_n\}$ is consistent. □

Definition 12 (Canonical Model)**.** *Given a signature $\alpha = \langle n, \mathcal{C}, \mathtt{Ap}\rangle$, the canonical model over α is the tuple $\mathfrak{G}^C = \langle n, \mathtt{Ac}^C, \mathcal{D}^C, S^C, R^C, \mathcal{V}^C\rangle$, where:*

- $\mathtt{Ac}^C = \mathcal{C} \cup \mathcal{C}^\star$;
- $\mathcal{D}^C = {\mathtt{Ac}^C}^n$;
- $S^C = \{X \mid X \text{ is a } \forall\text{-MCS over } \alpha^\star\}$;
- *for every $\vec{a} \in \mathcal{D}^C$, $\langle X, \vec{a}, Y\rangle \in R^C$ iff for every sentence φ we have that $\varphi \in Y$ implies $(\!(\vec{a})\!)\, \varphi \in X$;*
- *$X \in \mathcal{V}^C(p)$ iff $p \in X$ for all $p \in \mathtt{Ap}$.*

The proof of the next proposition is in [Catta *et al.*, 2025].

Proposition 6. *For all states $X, Y \in S^C$ and for every decision $\vec{a} \in \mathcal{D}^C$, it holds that $\langle X, \vec{a}, Y\rangle \in R^C$ iff for every sentence φ, $(\!(\vec{a})\!)\, \varphi \in X$ implies $\varphi \in Y$*

Now we are ready to show that $\mathfrak{G}^C$ is indeed a CGS (proof in [Catta *et al.*, 2025]), and then prove the Truth Lemma.

Proposition 7. *The canonical model $\mathfrak{G}^C$ is a CGS.*

Lemma 6 (Truth Lemma)**.** *For any state $X \in S^C$ and for any sentence φ, we have that $\mathfrak{G}^C, X \models \varphi$ iff $\varphi \in X$.*

Proof. The proof is by induction on φ. The *base case* $\varphi = p$ follows from the definition of $\mathcal{V}^C$. Boolean cases follow from the induction hypothesis (IH) and the properties of MCSs.

Case $\varphi = (\!(\vec{a})\!)\, \psi$. Let $\mathfrak{G}^C, X \models \varphi$. By the definition of semantics, this means that there is a Y such that $\langle X, \vec{a}, Y\rangle \in R^C$ and $\mathfrak{G}^C, Y \models \psi$. The latter is equivalent to $\psi \in Y$ by the IH, and by the definition of R^C we conclude that $\varphi \in X$.

Let $\varphi \in X$. By Lemma 5, there is a maximal consistent set of sentences Y over $\alpha^\star$ that has the $\forall$-property and such that $\{\psi\} \cup \{\theta \mid (\!(\vec{a})\!)\, \theta \in X\} \subseteq Y$. By Proposition 6 this means $\langle X, \vec{a}, Y\rangle \in R^C$, which, in conjunction with the fact that $\psi \in Y$, is equivalent to $\mathfrak{G}^C, X \models \varphi$ by the IH.

Case $\varphi = \forall x\psi$. If $\mathfrak{G}^C, X \models \varphi$, then, by the IH, it holds that (i) $\psi[a/x] \in X$ for every $a \in \mathtt{Ac}^C$. Now, assume towards a contradiction that $\varphi \notin X$. Since X is maximally consistent, we have that $\neg\forall x\psi \in X$. Moreover, since X has the $\forall$-property, there is a constant a such that $\psi[a/x] \rightarrow \forall x\psi \in X$. Then by (i) it follows that $\forall x\psi \in X$, which contradicts $\forall x\psi \notin X$.

Suppose that $\varphi \in X$, which implies, by axiom E and MP, that $\psi[a/x] \in X$ for every $a \in \mathtt{Ac}^C$. By the IH, we conclude that $\mathfrak{G}^C, X \models \varphi[a/x]$ for every $a \in \mathtt{Ac}^C$, which is equivalent to $\mathfrak{G}^C, X \models \forall x\varphi$ by the definition of semantics. □

We finally prove the completeness of FOCL.

Theorem 1. *For every set of formulae X and every formula φ, we have that $X \vdash \varphi$ iff $X \models \varphi$.*

Proof. Let $X \nvdash \varphi$. This means that $X \cup \{\neg\varphi\}$ is consistent, and, by Lemmas 2 and 3, there is a $\forall$-MCS Z, such that $X \cup \{\neg\varphi\} \subseteq Z$. As $\neg\varphi \in Z$, it holds that $\varphi \notin Z$, and by the truth lemma we have that $\mathfrak{G}^C, Z \models X$ and $\mathfrak{G}^C, Z \not\models \varphi$. □

5 Complexity Profile of FOCL

Now we turn to the complexity profile of FOCL, and show that the complexity of the model checking problem *PSPACE*-complete and that the satisfiability problem is undecidable.

Model Checking Let $\mathfrak{G} = \langle n, \mathtt{Ac}, \mathcal{D}, S, R, \mathcal{V}\rangle$ be a finite CGS, $s \in S$, and closed formula $\varphi \in$ FOCL constructed over a signature of $\mathfrak{G}$. The *local model checking problem* for FOCL consists in computing whether $\mathfrak{G}, s \models \varphi$.

Theorem 2. *The model checking problem for* FOCL *is PSPACE-complete.*

Proof. To show that the model checking problem for FOCL is in *PSPACE*, we provide an alternating recursive Algorithm 1[5]

[5]For brevity, we omit Boolean cases and the whole algorithm is available in [Catta *et al.*, 2025]

that takes as an input a finite CGS $\mathfrak{G}$, state of the CGS s, and a closed formula φ. The formula φ is provided in negation normal form (NNF), i.e. in equivalent rewriting, where all negations are pushed inside and appear only in front of propositional variables. To convert φ into the equivalent NNF formula, we can use propositional equivalences, interdefinability of quantifiers, and the validity $\neg(\!(\vec{t})\!)\,\varphi \leftrightarrow (\!(\vec{t})\!)\,\neg\varphi$. The size of a formula in NNF is at most linear in the size of the original formula. The correctness of the algorithm follows from the

Algorithm 1 An algorithm for model checking FOCL

```
procedure MC(𝔊, s, φ)
    case φ = ((a_1, ..., a_n)) ψ
        guess t ∈ S such that ⟨s, a_1, ..., a_n, t⟩ ∈ R
        return MC(𝔊, t, ψ)
    case φ = ∃xψ
        guess a ∈ Ac
        return MC(𝔊, s, ψ[a/x])
    case φ = ∀xψ
        universally choose a ∈ Ac
        return MC(𝔊, s, ψ[a/x])
end procedure
```

definition of the semantics. Its termination follows from the fact that every recursive call is run on a subformula of smaller size. Moreover, each call of the algorithm takes at most polynomial time, and hence it is in *APTIME*. From the fact that *APTIME* = *PSPACE* [Chandra *et al.*, 1981], we conclude that the model checking problem for FOCL is in *PSPACE*.

The hardness can be shown by the reduction from the satisfiability of quantified Boolean formulas (see [Catta *et al.*, 2025]). □

Remark 3. *The model checking problem for a related* SL *with simple goals (*SL[SG]*) is P-complete [Belardinelli* et al.*, 2019]. This is due to the fact that in* SL[SG] *the quantification prefix and the operators for assigning strategies to agents always go together. Hence, for example, the* FOCL *formula over two agents* $\theta := \forall x \exists y \forall z ((\!(z,x)\!)\,\varphi \wedge (\!(z,y)\!)\,\psi \wedge (\!(x,z)\!)\,\chi)$ *cannot be expressed in* SL[SG]*. The higher complexity of* FOCL *stems from the fact that quantifiers and strategy assignments are less rigid than in* SL[SG]*, and thus* FOCL *is closer to the full* SL *in this regard.*

Satisfiability Let $\varphi \in$ FOCL be a closed formula. The *satisfiability problem* for FOCL consists in determining whether there is a CGS $\mathfrak{G}, s$ such that $\mathfrak{G}, s \models \varphi$.

Theorem 3. *The satisfiability problem for* FOCL *is undecidable.*

The undecidability can be shown by employing the construction for SL from [Mogavero *et al.*, 2010; Mogavero *et al.*, 2017] using the reduction from the classic tiling problem [Wang, 1961]. In their construction, the authors use only formulae of the next-time fragment of SL, and the proof can be adapted for FOCL (see [Catta *et al.*, 2025] for details).

Remark 4. *A knowledgeable reader may point out that the proof in [Mogavero* et al.*, 2010; Mogavero* et al.*, 2017] employed the reduction from a more complex* recurring tiling *problem [Harel, 1983]. The problem is known to be Σ^1_1-complete, and this in particular implies that the next-time fragment of* SL*, and hence* FOCL*, is not recursively axiomatisable. This is at odds with the axiomatisation of* FOCL *presented in this paper. However, after a closer inspection, it turned out that the Σ^1_1-hardness proof provided in [Mogavero* et al.*, 2010; Mogavero* et al.*, 2017] is incomplete (though the standard, non-recurring, tiling construction stands, and hence the (standard) undecidability). As of now, no fix to this problem has been presented, and hence the existence of a recursive axiomatisation of* SL *is now an open question. Note, again, that the (standard) undecidability still holds*[6].

6 Discussion

We introduced *first-order coalition logic* (FOCL), which combines features of both CL and SL, and, additionally, allows for explicit action labels in the syntax. With FOCL we have solved several exciting problems. First, we showed that it is strictly more expressive than other known CL's, and that its model checking problem is *PSPACE*-complete. We then also argued that the satisfiability problem for FOCL is undecidable, pointing out an incomplete result in the foundational SL paper [Mogavero *et al.*, 2010] and thus reopening the question of whether SL is recursively axiomatisable. Moreover, we provided a sound and complete axiomatisation of FOCL. This is significant, since, to the best of our knowledge, it is *the first axiomatisation of any strategy logic*.

There is a plethora of open research questions that one can tackle building on our work. Perhaps the most immediate one is finding an axiomatisation of an extension of FOCL with LTL modalities. In such a way, we would be able to advance towards axiomatisations of such rich fragments of SL as *one-goal* SL [Mogavero *et al.*, 2017] and *flat conjunctive-goal* SL [Acar *et al.*, 2019]. It is also quite interesting to consider FOCL in the context of imperfect information (see [Ågotnes *et al.*, 2015] for an overview).

While dealing with the undedicability of FOCL, we mentioned the next-time fragment of SL. To the best of our knowledge, such a fragment has never been singled out and studied before. Hence, it is tempting to look into the variations of this fragment[7], identify the axiomatisable ones, and have a proper comparison of the latter with FOCL.

As STIT logics [Horty, 2001] admit CGS semantics [Boudou and Lorini, 2018; Broersen and Herzig, 2015], another avenue of exciting further research is establishing the exact relation between FOCL and variants of STIT logics like *group* STIT [Herzig and Schwarzentruber, 2008; Lorini and Schwarzentruber, 2011].

[6] The gap in the proof of non-axiomatisability of SL was acknowledged and corroborated by the authors of [Mogavero *et al.*, 2010; Mogavero *et al.*, 2017] in personal communication. The general sentiment is that SL is still not recursively axiomatisable, but to show this result, one will have to employ richer features of SL, beyond its next-time fragment.

[7] There is a nuance in how we can define the next-time fragment of SL. Two obvious candidates are the fragment, where every neXt modality is immediately preceded by an assignment (similar to ATL), and the fragment, where we do not have such a condition (similar to ATL*).

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